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Impressed sources and fields in the volume-integral-equation formulation of electromagnetic scattering by a finite object: A tutorial

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ABSTRACT

Although free space cannot generate electromagnetic waves, the majority of existing accounts of frequency-domain electromagnetic scattering by particles and particle groups are based on the postulate of existence of an impressed incident field, usually in the form of a plane wave. In this tutorial we discuss how to account for the actual existence of impressed source currents rather than impressed incident fields. Specifically, we outline a self-consistent theoretical formalism describing electromagnetic scattering by an arbitrary finite object in the presence of arbitrarily distributed impressed currents, some of which can be far removed from the object and some can reside in its vicinity, including inside the object. To make the resulting formalism applicable to a wide range of scattering-object morphologies, we use the framework of the volume integral equation formulation of electromagnetic scattering, couple it with the notion of the transition operator, and exploit the fundamental symmetry property of this operator. Among novel results, this tutorial includes a streamlined proof of fundamental symmetry (reciprocity) relations, a simplified derivation of the Foldy equations, and an explicit analytical expression for the transition operator of a multi-component scattering object.

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1. Introduction

Standard theoretical accounts of the frequency-domain scattering of electromagnetic waves by particles and particle groups in the framework of macroscopic Maxwell's electrodynamics have commonly been based on assuming the existence of an impressed incident field (see, e.g., Refs. [1–19]). The designation "impressed" (or "enforced") implies that the incident field is independent of the total electromagnetic field existing in the presence of a scattering object. The standard line of thought has been that the scattering problem can be formulated assuming that the entire space is devoid of source currents and can be reduced to considering an inhomogeneous differential equation whose solution consists of two parts. The first one is a solution of the corresponding homogeneous equation, while the second one is a physically appropriate solution of the inhomogeneous equation. The former implies the absence of the scattering object and thus represents the impressed incident field in the form of an *ad hoc* free-space solution of the macroscopic Maxwell equations. More often than not, the incident field has been postulated to be a plane electromagnetic wave, although other types of wave have increasingly been considered [20].

Such treatments obscure the inescapable fact that electromagnetic waves cannot be generated by a free space, and so there must be an actual source of the incident field. An advanced description of the emission of electromagnetic waves by elementary charges is supplied by quantum electrodynamics (QED) [21–27]. However, neither the QED treatment nor even the classical microscopic Maxwell–Lorentz electrodynamics [28–33] can realistically be applied to the analysis of electromagnetic scattering by exceedingly complex macroscopic objects consisting of an enormous number of elementary charges. Hence the widespread use of macroscopic Maxwell's electromagnetics [34–38] based on the premise that electromagnetic waves are created by macroscopic charge currents. It would therefore be highly desirable to have a self-consistent scattering formalism explicitly built on the existence of impressed (or enforced) source currents rather than impressed incident fields. The designation "impressed" again serves to indicate that unlike the secondary conducting currents, the primary source currents are independent of the total electromagnetic field existing in the presence of a scattering object.

The standard treatment based on the assumption of the existence of an impressed incident field in the form of a plane electro-

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magnetic wave is sometimes rationalized by referring to a situation wherein a point-like source current is located at an exceedingly large distance from a finite scattering object (e.g., Ref. [37]). It can indeed be shown that in this case the free-space field generated by the current in the vicinity of the object is an outgoing spherical electromagnetic wave that can be considered a locally plane wave owing to the smallness of the solid angle subtended by the object as viewed from the remote source region. There are situations, however, when it is appropriate to assume the simultaneous existence of both remote and local impressed source currents, including those residing inside the scattering object. Such situations arise, e.g., in the context of the semi-classical fluctuational electrodynamics which treats thermal emission of electromagnetic waves as being the result of impressed fluctuating currents inside an object having a positive absolute temperature [39–58]. If the object is also subjected to external illumination then one must explicitly solve the scattering problem featuring both remote time-harmonic and local fluctuating impressed currents. A typical geophysical example is a cloud of particles in a planetary atmosphere which can both be illuminated by the incident stellar light at near-infrared wavelengths and emit its own near-infrared radiation [59–63].

Consistent with the above discussion, the main objective of this tutorial is to outline a maximally general and self-consistent theoretical formalism describing electromagnetic scattering by an arbitrary finite object in the presence of arbitrarily distributed impressed currents, some of which can be far removed from the object and some can reside in its vicinity, including inside the object. To explicitly allow for internal inhomogeneity of the scattering object, we use the framework of the volume integral equation (VIE) formulation of electromagnetic scattering, couple it with the notion of the transition operator, and exploit the fundamental symmetry property of this operator. Some aspects of the resulting formalism are scattered over the existing literature and some represent a relatively straightforward extension of previously published results (see, e.g., Refs. [11,42,44,54,57,64–68]). Yet given the compactness, uniformity, and generality of this formalism, it appears worthwhile to summarize it in the form of a self-contained stand-alone tutorial. Moreover, we also discuss some novel results, including an elegant proof of fundamental symmetry (reciprocity) relations, a streamlined derivation of the Foldy equations, and an explicit expression of the transition operator of a multi-component object in terms of the corresponding individual-component transition operators.

2. Scattering problem

The derivation of the macroscopic Maxwell equations from either the QED or the microscopic Maxwell–Lorentz equations is still incomplete [69]. Therefore, we will invoke the former essentially as a set of phenomenological axioms [34–38]. In what follows, we imply the monochromatic $\exp(-\mathrm{i}\omega t)$ dependence of all fields and sources, where t is time, ω is the angular frequency, and $\mathrm{i}=(-1)^{1/2}$. Consider a fixed finite object embedded in an infinite medium that is assumed to be homogeneous, linear, isotropic, and potentially absorbing. Accordingly, the complex permittivity of the host medium ε_1 can have a non-zero imaginary part: $0 \le \arg(\varepsilon_1) < \pi$. The object can be either a single connected body or a cluster consisting of a finite number N of non-overlapping connected components; it occupies collectively an "interior" region V_{INT} defined by

$$V_{\mathrm{INT}} \triangleq \bigcup_{i=1}^{N} V_{\mathrm{INT}}^{i}, \tag{1}$$

where V_{INT}^{i} is the volume occupied by the ith component (see Fig. 1). The object is surrounded by the infinite exterior domain V_{EXT} such that $V_{\mathrm{INT}} \cup V_{\mathrm{EXT}} = \mathbb{R}^3$, where $\mathbb{R}^3$ is the entire three-

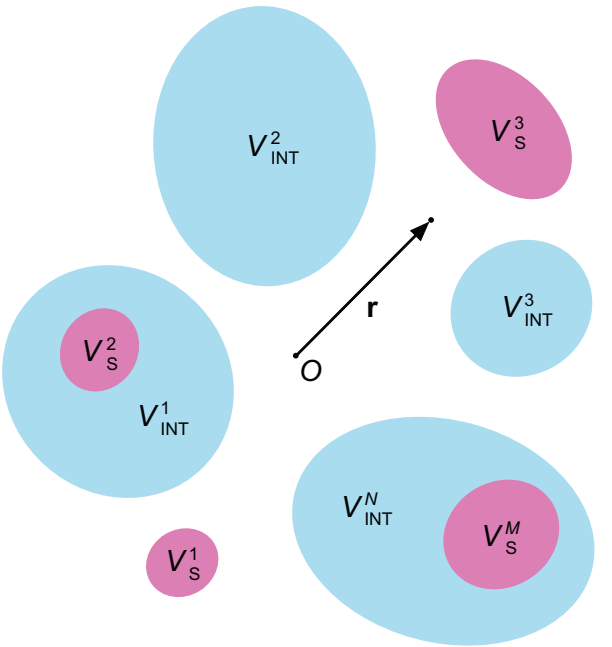

Fig. 1. Schematic representation of the scattering problem.

dimensional space. The interior region is filled with isotropic, linear, and possibly inhomogeneous materials. Point O serves as the common origin of all position vectors.

We assume that in addition to the object, there are $0 \le M < \infty$ impressed source currents occupying finite connected non-overlapping volumes V_{S}^{i}. As explained above, the term "impressed" means that these source currents are unaffected by the resulting electromagnetic field existing in the presence of the scattering object.

Finally, we assume that both the infinite host medium and the finite scattering object are non-magnetic. Then the frequency-domain Maxwell curl equations for the monochromatic electromagnetic field can be written in SI units as follows:

$$\left.\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \mathrm{i}\omega\mu_0 \mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -\mathrm{i}\omega\varepsilon_1 \mathbf{E}(\mathbf{r}) + \mathbf{J}^{\mathrm{S}}(\mathbf{r}) \end{aligned}\right\} \mathbf{r} \in V_{\mathrm{EXT}}, \tag{2}$$

$$\left.\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \mathrm{i}\omega\mu_0 \mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -\mathrm{i}\omega\varepsilon_2(\mathbf{r}) \mathbf{E}(\mathbf{r}) + \mathbf{J}^{\mathrm{S}}(\mathbf{r}) \end{aligned}\right\} \mathbf{r} \in V_{\mathrm{INT}}, \tag{3}$$

where $\mathbf{E}(\mathbf{r})$ is the electric and $\mathbf{H}(\mathbf{r})$ the magnetic field; μ_0 is the magnetic permeability of a vacuum; $\varepsilon_2(\mathbf{r})$ is the complex permittivity of the object; and $\mathbf{J}^{\mathrm{S}}(\mathbf{r})$ is the impressed source current. We assume for simplicity that $\varepsilon_2(\mathbf{r})$ is a sufficiently smooth function of $\mathbf{r}$ inside each V_{INT}^{i} so that the interior of V_{INT}^{i} contains no sharp optical interfaces. It is also clear that

$$\mathbf{J}^{\mathrm{S}}(\mathbf{r}) = \mathbf{0} \text{ if } \mathbf{r} \notin V_{\mathrm{S}}, \tag{4}$$

where $\mathbf{0}$ is a zero vector and V_{S} is the total volume occupied by the impressed sources:

$$V_{\mathrm{S}} \triangleq \bigcup_{i=1}^{M} V_{\mathrm{S}}^{i}. \tag{5}$$

The corresponding boundary conditions read:

$$\left.\begin{aligned} \hat{\mathbf{n}} \times [\mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r})] &= \mathbf{0} \\ \hat{\mathbf{n}} \times [\mathbf{H}_1(\mathbf{r}) - \mathbf{H}_2(\mathbf{r})] &= \mathbf{0} \end{aligned}\right\} \mathbf{r} \in S_{\mathrm{INT}}, \tag{6}$$

where the subscripts 1 and 2 correspond to the exterior and interior sides of the boundary S_{INT} of the object, respectively, and $\hat{\mathbf{n}}$ is the local outward normal to S_{INT}. According to Eq. (1), S_{INT} is the

union of the closed surfaces of the N connected components of the object:

$$S_{\rm INT} = \bigcup_{i=1}^{N} S_{\rm INT}^{i}. \tag{7}$$

For simplicity, we assume $S_{\rm INT}$ to be sufficiently smooth (e.g., lacking sharp edges and corners).

To guarantee the uniqueness of solution of the general scattering problem [70–73], we postulate that besides satisfying the boundary conditions (6), the total electromagnetic field satisfies the following condition at infinity:

$$\lim_{r\to\infty}\left[\sqrt{\mu_0}\,\mathbf{r}\times\mathbf{H}(\mathbf{r})+r\sqrt{\varepsilon_1}\,\mathbf{E}(\mathbf{r})\right]=\mathbf{0}, \tag{8}$$

where $r=|\mathbf{r}|$ is the distance from the origin to the observation point (Fig. 1). The limit (8) holds uniformly over all outgoing directions $\hat{\mathbf{r}}=\mathbf{r}/r$ and is traditionally called the Sommerfeld [74,75] or Silver–Müller [2,76] radiation condition.

The above-formulated scattering problem is well defined mathematically and, as stated, encompasses a broad range of scenarios in terms of the morphology of the source regions and that of the scattering object. Yet it could be criticized for not specifying explicitly what physical mechanisms are responsible for the impressed sources. This criticism may be especially relevant in the case of impressed source currents residing inside the object (see, e.g., the discussion in Ref. [77]). Microphysically this implies that the same elementary charges contribute to the impressed currents as well as to the secondary conducting currents. As we have mentioned in the Introduction, this assumption is at the very heart of fluctuational electrodynamics. Nevertheless, it should be recognized that in some cases the model of internal impressed source currents may be pushing the classical macroscopic electromagnetics beyond its conceptual realm and may require an explicit derivation from quantum physics. A relevant mathematical issue can be the interchange of the order of averaging over a microscopic volume in the definition of both macroscopic fields and impressed currents.

3. Modified volume integral equation

Eqs. (2) and (3) demonstrate that if $\mathbf{E}(\mathbf{r})$ is known everywhere in space then $\mathbf{H}(\mathbf{r})$ can also be determined everywhere in space. We will therefore focus on the derivation of the modified VIE for the electric field only.

Eqs. (2) and (3) imply the following vector wave equations for $\mathbf{E}(\mathbf{r})$:

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r})-k_1^2\mathbf{E}(\mathbf{r})=\mathrm{i}\omega\mu_0\mathbf{J}^{\rm S}(\mathbf{r}),\ \mathbf{r}\in V_{\rm EXT}, \tag{9}$$

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r})-\omega^2\varepsilon_2(\mathbf{r})\mu_0\mathbf{E}(\mathbf{r})=\mathrm{i}\omega\mu_0\mathbf{J}^{\rm S}(\mathbf{r}),\ \mathbf{r}\in V_{\rm INT}. \tag{10}$$

These two equations can be rewritten as a single inhomogeneous differential equation

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r})-k_1^2\mathbf{E}(\mathbf{r})=\mathbf{j}(\mathbf{r})+\mathrm{i}\omega\mu_0\mathbf{J}^{\rm S}(\mathbf{r}),\ \mathbf{r}\in\mathbb{R}^3, \tag{11}$$

where

$$\mathbf{j}(\mathbf{r})\triangleq U(\mathbf{r})\mathbf{E}(\mathbf{r}) \tag{12}$$

is the *forcing function*,

$$U(\mathbf{r})\triangleq\begin{cases}0,\ \mathbf{r}\in V_{\rm EXT},\\ \omega^2\varepsilon_2(\mathbf{r})\mu_0-k_1^2,\ \mathbf{r}\in V_{\rm INT}\end{cases} \tag{13}$$

is the potential function, and

$$k_1\triangleq\omega\sqrt{\varepsilon_1\mu_0} \tag{14}$$

is the wave number in the host medium. A key property of the forcing function is that it vanishes everywhere outside the finite interior region $V_{\rm INT}$.

We can now exploit the fact that any solution of the inhomogeneous linear differential equation (11) can be expressed as a sum of three parts: (i) a solution of the respective homogeneous equation with the right-hand side identically equal to zero; (ii) a particular solution of the inhomogeneous equation

$$\nabla\times\nabla\times\mathbf{E}^{\rm S}(\mathbf{r})-k_1^2\mathbf{E}^{\rm S}(\mathbf{r})=\mathrm{i}\omega\mu_0\mathbf{J}^{\rm S}(\mathbf{r}),\ \mathbf{r}\in\mathbb{R}^3; \tag{15}$$

and (iii) a particular solution of the inhomogeneous equation

$$\nabla\times\nabla\times\mathbf{E}^{\rm sca}(\mathbf{r})-k_1^2\mathbf{E}^{\rm sca}(\mathbf{r})=\mathbf{j}(\mathbf{r}),\ \mathbf{r}\in\mathbb{R}^3. \tag{16}$$

The first part defines the field that would exist in free space in the absence of the object and of the impressed sources. Based on physical grounds, it is postulated to be equal to zero. In other words, we do not consider artificial impressed incident fields propagating from infinity (see also the discussion below).

The second part corresponds to the situation with no scattering object present. The physically appropriate particular solution of Eq. (15) satisfying the radiation condition at infinity is well known [37,78,79]:

$$\mathbf{E}^{\rm S}(\mathbf{r})=\mathrm{i}\omega\mu_0\int_{V_{\rm S}}\mathrm{d}^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}^{\rm S}(\mathbf{r}'), \tag{17}$$

where

$$\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')=\left(\overleftrightarrow{I}+\frac{1}{k_1^2}\nabla\otimes\nabla\right)g(\mathbf{r},\mathbf{r}') \tag{18}$$

is the free-space dyadic Green's function, $\overleftrightarrow{I}$ is the identity (or unit) dyadic, $\otimes$ denotes the dyadic product of two vectors, and

$$g(\mathbf{r},\mathbf{r}')=\frac{\exp\left(\mathrm{i}k_1|\mathbf{r}-\mathbf{r}'|\right)}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{19}$$

is the scalar Green's function. $\mathbf{E}^{\rm S}(\mathbf{r})$ can be referred to as the free-space source-generated field.

The third part is the scattered field $\mathbf{E}^{\rm sca}(\mathbf{r})$ corresponding to the forcing function $\mathbf{j}(\mathbf{r})$ and satisfying the boundary conditions (6) as well as the radiation condition (8).

We can now precisely follow the line of the derivation detailed in Section 4.3 of Ref. [19] and deduce:

$$\mathbf{E}^{\rm sca}(\mathbf{r})=\int_{V_{\rm INT}}\mathrm{d}^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{j}(\mathbf{r}'),\ \mathbf{r}\in\mathbb{R}^3. \tag{20}$$

It is important to note that strictly speaking, Eqs. (17) and (20) imply a non-integrable singularity when $\mathbf{r}\in V_{\rm INT}$ and/or $\mathbf{r}\in V_{\rm S}$. The standard implicit remedy [37,80,81] is to assume that the integration is carried in the following specific principal-value sense:

$$\int_V\mathrm{d}^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{F}(\mathbf{r}')=\lim_{V_0\to0}\int_{V\setminus V_0}\mathrm{d}^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{F}(\mathbf{r}')-\frac{1}{3k_1^2}\mathbf{F}(\mathbf{r}), \tag{21}$$

where V_0 is a spherical exclusion volume around $\mathbf{r}$. This aspect of the VIE formalism will be further discussed in the concluding section.

The final step is to substitute Eq. (12) in Eq. (20), which yields

$$\mathbf{E}(\mathbf{r})=\mathbf{E}^{\rm S}(\mathbf{r})+\mathbf{E}^{\rm sca}(\mathbf{r}), \tag{22}$$

where the scattered field is given by

$$\mathbf{E}^{\rm sca}(\mathbf{r})=\int_{V_{\rm INT}}\mathrm{d}^3\mathbf{r}'\,U(\mathbf{r}')\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}'). \tag{23}$$

Eqs. (17), (22), and (23) yield collectively the sought modification of the conventional VIE explicitly accounting for the impressed source currents rather than impressed incident fields:

$$\mathbf{E}(\mathbf{r}) = \mathrm{i}\omega\mu_0 \int_{V_S} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}^S(\mathbf{r}') + \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}' U(\mathbf{r}')\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}'),\ \mathbf{r}\in\mathbb{R}^3. \tag{24}$$

The only formal difference of the modified VIE from its conventional counterpart [19] is that the impressed incident field $\mathbf{E}^{\mathrm{inc}}(\mathbf{r})$ in the latter has been replaced by the free-space source-generated field $\mathbf{E}^S(\mathbf{r})$ in the former. This result serves to provide justification to the common practice of "hiding" impressed sources by assuming that the impressed "incident" field they could generate is known *a priori*.

Finally we note that owing to Eqs. (4) and (13), the integration domains in Eqs. (17), (23), and (24) can formally be extended to cover the entire space $\mathbb{R}^3$.

4. Transition dyadic

As usual, we assume that the linearity of and the specific integration domain in Eq. (23) imply the possibility of expressing the scattered electric field linearly in terms of the source-generated field inside V_{INT}. In other words, we assume that for any $\mathbf{E}^S(\mathbf{r})$, $U(\mathbf{r})\mathbf{E}(\mathbf{r})$ can be expressed in terms of $\mathbf{E}^S(\mathbf{r})$ via a linear integral operator (called the transition operator)

$$U(\mathbf{r})\mathbf{E}(\mathbf{r}) = \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{T}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^S(\mathbf{r}'),\ \mathbf{r}\in V_{\mathrm{INT}} \tag{25}$$

whose kernel $\overleftrightarrow{T}(\mathbf{r},\mathbf{r}')$ is called the transition dyadic. This quantity was first introduced in the framework of electromagnetic scattering by Tsang and Kong [65]. Substituting this expression in Eq. (23) yields

$$\mathbf{E}^{\mathrm{sca}}(\mathbf{r}) = \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}'' \overleftrightarrow{T}(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}^S(\mathbf{r}''). \tag{26}$$

Eqs. (22), (23), and (26) then imply that $\overleftrightarrow{T}(\mathbf{r},\mathbf{r}')$ satisfies the following Lippmann–Schwinger integral equation:

$$\overleftrightarrow{T}(\mathbf{r},\mathbf{r}') = U(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\overleftrightarrow{I} + U(\mathbf{r})\int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}'' \overleftrightarrow{G}(\mathbf{r},\mathbf{r}'')\cdot\overleftrightarrow{T}(\mathbf{r}'',\mathbf{r}'). \tag{27}$$

A fundamental property of the transition dyadic explicit in Eq. (27) is that it is fully defined by the scattering object alone (i.e., by the spatial distribution of the electric permittivity throughout V_{INT}) and is completely independent of the impressed sources. In other words, $\overleftrightarrow{T}(\mathbf{r},\mathbf{r}')$ serves as a unique and complete "scattering identifier" of the object.

Eq. (27) defines the transition dyadic only inside the scattering object. It is convenient to complete the definition by assuming that

$$\overleftrightarrow{T}(\mathbf{r},\mathbf{r}') = \overleftrightarrow{0}\ \text{unless}\ \mathbf{r}\in V_{\mathrm{INT}}\ \text{and}\ \mathbf{r}'\in V_{\mathrm{INT}}, \tag{28}$$

where $\overleftrightarrow{0}$ is a zero dyad. This implies that the integration domain in Eqs. (26) and (27) can formally be extended to the entire space $\mathbb{R}^3$.

5. Additivity

Based on the fundamental principle of superposition (i.e., the linearity of the electromagnetic scattering problem), we can expect the additivity of the total fields generated individually by different impressed sources. Indeed, the individual total field generated by the ith source region V_S^i in the absence of all the other sources is the solution of the VIE

$$\mathbf{E}^i(\mathbf{r}) = \mathrm{i}\omega\mu_0 \int_{V_S^i} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}^S(\mathbf{r}') + \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}' U(\mathbf{r}')\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^i(\mathbf{r}'). \tag{29}$$

Summing up these M fields yields Eq. (24) in which

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{M} \mathbf{E}^i(\mathbf{r}). \tag{30}$$

It is easily seen that the additivity of the total fields generated by different source regions is also implicit in Eq. (26).

6. Source Green's dyadic

Eqs. (17), (22), and (26) imply that the total field can be expressed in terms of the impressed sources $\mathbf{J}^S(\mathbf{r})$ according to

$$\mathbf{E}(\mathbf{r}) = \mathrm{i}\omega\mu_0 \int_{V_S} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{G}_S(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}^S(\mathbf{r}'), \tag{31}$$

where

$$\overleftrightarrow{G}_S(\mathbf{r},\mathbf{r}') \triangleq \overleftrightarrow{G}(\mathbf{r},\mathbf{r}') + \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}'' \overleftrightarrow{G}(\mathbf{r},\mathbf{r}'') \cdot \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}''' \overleftrightarrow{T}(\mathbf{r}'',\mathbf{r}''')\cdot\overleftrightarrow{G}(\mathbf{r}''',\mathbf{r}') \tag{32}$$

is the source Green's dyadic.

It is seen that like the transition dyadic, $\overleftrightarrow{G}_S(\mathbf{r},\mathbf{r}')$ is completely independent of the impressed sources. Furthermore, Eq. (32) shows that $\overleftrightarrow{G}_S(\mathbf{r},\mathbf{r}')$ is defined everywhere in $\mathbb{R}^3$, which eliminates the need to define separate source Green's dyadics by considering points $\mathbf{r}$ and $\mathbf{r}'$ in pairs of specific domains.

It is easy to verify that Eqs. (24) and (31) yield the following closed-form integral equation for the source Green's dyadic [65]:

$$\overleftrightarrow{G}_S(\mathbf{r},\mathbf{r}') = \overleftrightarrow{G}(\mathbf{r},\mathbf{r}') + \int_{V_{\mathrm{INT}}} \mathrm{d}^3\mathbf{r}'' U(\mathbf{r}'')\overleftrightarrow{G}(\mathbf{r},\mathbf{r}'')\cdot\overleftrightarrow{G}_S(\mathbf{r}'',\mathbf{r}'). \tag{33}$$

As before, the integration domains in Eqs. (32) and (33) can formally be extended to encompass the entire space.

7. Short-hand integral-operator notation

Let us define the potential dyadic

$$\overleftrightarrow{U}(\mathbf{r},\mathbf{r}') \triangleq U(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\overleftrightarrow{I}, \tag{34}$$

where $\delta(\mathbf{r})$ is the three-dimensional delta function, and introduce short-hand integral-operator notation according to

$$\hat{B}E \triangleq \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{r}' \overleftrightarrow{B}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}') \tag{35}$$

and

$$\left(\hat{B}\hat{C}\right)E \triangleq \hat{B}\left(\hat{C}E\right) \tag{36}$$

(note that this use of a caret above an Italic character to denote an integral operator should not be confused with a caret above a bold-face upright character denoting a unit vector). Then the main formulas of the preceding section can be re-written as follows:

$$E^S = \mathrm{i}\omega\mu_0\hat{G}J^S \quad (\text{cf. Eq. (17)}) \tag{37}$$

$$E = \mathrm{i}\omega\mu_0\hat{G}J^S + \hat{G}\hat{U}E = E^S + \hat{G}\hat{U}E \quad (\text{cf. Eq.(24)}), \tag{38}$$

$$\hat{U}E = \hat{T}E^S \ (\text{cf. Eq. (25)}), \tag{39}$$

$$E = E^{\mathrm{S}} + \hat{G}\hat{T}E^{\mathrm{S}} \ \text{(cf. Eqs. (22) and (26))}, \tag{40}$$

$$\hat{T} = \hat{U} + \hat{U}\hat{G}\hat{T} \ \ \text{(cf. Eq. (27))}, \tag{41}$$

$$E = \mathrm{i}\omega\mu_0\hat{G}_{\mathrm{S}}J^{\mathrm{S}} \ \text{(cf. Eq. (31))}, \tag{42}$$

$$\hat{G}_{\mathrm{S}} = \hat{G} + \hat{G}\hat{T}\hat{G} \ \text{(cf. Eq. (32))}, \tag{43}$$

and

$$\hat{G}_{\mathrm{S}} = \hat{G} + \hat{G}\hat{U}\hat{G}_{\mathrm{S}} \ \text{(cf. Eq. (33))}, \tag{44}$$

where the order of the operators is essential and cannot be changed arbitrarily.

Let us now define the *pseudo-adjoint* (transposed) integral operator $\hat{B}^{\mathrm{t}}$ according to Ref. [68]:

$$\hat{B}^{\mathrm{t}}E \triangleq \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{r}' \left[\overleftrightarrow{B}(\mathbf{r}', \mathbf{r})\right]^{\mathrm{T}} \cdot \mathbf{E}(\mathbf{r}'), \tag{45}$$

where T denotes the conventional transpose of a dyadic. It is analogous to the (Hermitian) adjoint operator $\hat{B}^*$ but is based on the pseudo inner product instead of the standard inner product of the vector space $L^2(\mathbb{R}^3)^3$ [68]. Alternatively, $\hat{B}^{\mathrm{t}}$ can be defined as

$$\hat{B}^{\mathrm{t}} \triangleq \hat{Z}\hat{B}^*\hat{Z}, \tag{46}$$

where $\hat{Z}$ is the antilinear operator performing pointwise complex conjugation [82]. In particular $\hat{Z}\hat{Z} = \hat{I}$, where the unit integral operator $\hat{I}$ has the dyadic $\delta(\mathbf{r} - \mathbf{r}')\overleftrightarrow{I}$ as its kernel.

Taking pseudo adjoint of an operator has many properties analogous to those of matrix or dyadic transposition as well as those of the Hermitian adjoint. In particular, the dyadic identity

$$\left(\overleftrightarrow{B}\overleftrightarrow{C}\right)^{\mathrm{T}} = \overleftrightarrow{C}^{\mathrm{T}}\overleftrightarrow{B}^{\mathrm{T}} \tag{47}$$

(see, e.g., Appendix A in Ref. [19]) and Eq. (45) yield

$$\left(\hat{B}\hat{C}\right)^{\mathrm{t}} = \hat{C}^{\mathrm{t}}\hat{B}^{\mathrm{t}}, \tag{48}$$

which in turn implies

$$\left(\hat{B}^{\mathrm{t}}\right)^{-1} = \left(\hat{B}^{-1}\right)^{\mathrm{t}}, \tag{49}$$

where one inverse exists and is bounded if and only if the other one exists and is bounded. Eq. (49) is derived by a direct multiplication of $\hat{B}^{\mathrm{t}}$ by $(\hat{B}^{-1})^{\mathrm{t}}$ in both possible orders (see also Ref. [68]).

We call an integral operator *pseudo self-adjoint* if it is equal to its own pseudo adjoint, $\hat{B} = \hat{B}^{\mathrm{t}}$, which is equivalent to its dyadic kernel being (*complex*) *symmetric*, i.e., satisfying

$$\overleftrightarrow{B}(\mathbf{r}, \mathbf{r}') = \left[\overleftrightarrow{B}(\mathbf{r}', \mathbf{r})\right]^{\mathrm{T}}, \tag{50}$$

which includes the interchange of the arguments as well as dyadic transposition. Note that pseudo self-adjoint is the simplest case of a "complex symmetric" linear operator defined in Ref. [82] for a specific choice of $\hat{Z}$; hence we use the former term to avoid ambiguity.

Importantly, the free-space dyadic Green's operator $\hat{G}$ is pseudo self-adjoint, and the same is obviously true of $\hat{U}$ for isotropic (as considered in this tutorial) or, more generally, for any reciprocal media [83].

As discussed in Section 3, we assume that the VIE (38) has a solution, and this solution is unique. This implies that the operator $\hat{I} - \hat{G}\hat{U}$ is (uniquely) invertible. The invertibility of the operator $\hat{I} - \hat{G}\hat{U}$ ensures, in turn, the existence of a unique solution of Eq. (44). The operator $\hat{I} - \hat{U}\hat{G}$ is the pseudo adjoint of $\hat{I} - \hat{G}\hat{U}$ (as a consequence of the pseudo self-adjointness of the multiplicative factors and Eq. (48)) and, according to Eq. (49), is also invertible, thereby ensuring the existence of a unique solution of Eq. (41).

8. Symmetry of the transition dyadic and of the source Green's dyadic

A fundamental and useful property of the transition dyadic is the symmetry relation which follows from the pseudo self-adjointness of the corresponding integral operator [57]:

$$\overleftrightarrow{T}(\mathbf{r}, \mathbf{r}') = \left[\overleftrightarrow{T}(\mathbf{r}', \mathbf{r})\right]^{\mathrm{T}}, \ \mathbf{r}, \mathbf{r}' \in \mathbb{R}^3. \tag{51}$$

A somewhat informal way to infer Eq. (51) is to iterate Eq. (41):

$$\hat{T} = \hat{U} + \hat{U}\hat{G}\hat{U} + \hat{U}\hat{G}\hat{U}\hat{G}\hat{U} + \cdots. \tag{52}$$

Since each term in this expansion is pseudo self-adjoint, their sum must also be a pseudo self-adjoint operator. However, there is no guarantee that the series (52) always converges, which calls for a more formal proof of the symmetry relation (51).

Taking pseudo adjoint of both sides of Eq. (41), recalling Eq. (48), and accounting for the pseudo self-adjointness of $\hat{U}$ and $\hat{G}$ yields

$$\hat{T}^{\mathrm{t}} = \hat{U} + \hat{T}^{\mathrm{t}}\hat{G}\hat{U}. \tag{53}$$

Upon left-multiplying this formula by $\hat{I} - \hat{U}\hat{G}$ we have

$$\hat{T}^{\mathrm{t}} - \hat{U}\hat{G}\hat{T}^{\mathrm{t}} = \hat{U} - \hat{U}\hat{G}\hat{U} + \hat{T}^{\mathrm{t}}\hat{G}\hat{U} - \hat{U}\hat{G}\hat{T}^{\mathrm{t}}\hat{G}\hat{U} \tag{54}$$

or

$$\left(\hat{T}^{\mathrm{t}} - \hat{U} - \hat{U}\hat{G}\hat{T}^{\mathrm{t}}\right)\left(\hat{I} - \hat{G}\hat{U}\right) = \hat{0}, \tag{55}$$

where $\hat{0}$ is a zero operator. The invertibility of the operator $\hat{I} - \hat{G}\hat{U}$ discussed in the preceding section implies

$$\hat{T}^{\mathrm{t}} = \hat{U} + \hat{U}\hat{G}\hat{T}^{\mathrm{t}}. \tag{56}$$

Thus $\hat{T}^{\mathrm{t}}$ satisfies the same Eq. (41) as $\hat{T}$. Since this equation has a unique solution, we arrive at the symmetry relation

$$\hat{T} = \hat{T}^{\mathrm{t}}, \tag{57}$$

that is, Eq. (51).

It is also easily shown that Eq. (43) coupled with the symmetry relation (57), the pseudo self-adjointness of the free-space Green's operator, and Eq. (48) implies the pseudo self-adjointness of the source Green's operator:

$$\hat{G}_{\mathrm{S}} = \hat{G}_{\mathrm{S}}^{\mathrm{t}}, \tag{58}$$

or, in terms of the source Green's dyadic,

$$\overleftrightarrow{G}_{\mathrm{S}}(\mathbf{r}, \mathbf{r}') = \left[\overleftrightarrow{G}_{\mathrm{S}}(\mathbf{r}', \mathbf{r})\right]^{\mathrm{T}}, \ \mathbf{r}, \mathbf{r}' \in \mathbb{R}^3. \tag{59}$$

It is in fact remarkable how the straightforward use of the pseudo self-adjointness of the transition operator allows one to completely bypass the cumbersome use of the Lorentz reciprocity in the derivation of the symmetry of the source Green's dyadic (cf. Ref. [44]). The relation between reciprocity and pseudo self-adjointness of integral operators was discussed in Ref. [68], but without an explicit derivation for an arbitrary scattering object.

9. Far-field point-like source

To give an instructive example of applying Eqs. (17) and (26), let us consider a point-like source located in the far zone of the entire volume V_{INT} and centered at $\mathbf{r}_{\mathrm{S}}$ (see Fig. 2):

$$\mathbf{J}^{\mathrm{S}}(\mathbf{r}'') = \delta(\mathbf{r}'' - \mathbf{r}_{\mathrm{S}})\mathbf{J}_0^{\mathrm{S}}. \tag{60}$$

To this end, we use the asymptotic formula

$$\overleftrightarrow{G}(\mathbf{r}'', \mathbf{r}_{\mathrm{S}}) \to \left(\overleftrightarrow{I} - \hat{\mathbf{r}}_{\mathrm{S}} \otimes \hat{\mathbf{r}}_{\mathrm{S}}\right)\frac{\exp(\mathrm{i}k_1 r_{\mathrm{S}})}{4\pi r_{\mathrm{S}}}\exp\left(-\mathrm{i}k_1\hat{\mathbf{r}}_{\mathrm{S}}\cdot\mathbf{r}''\right) \tag{61}$$

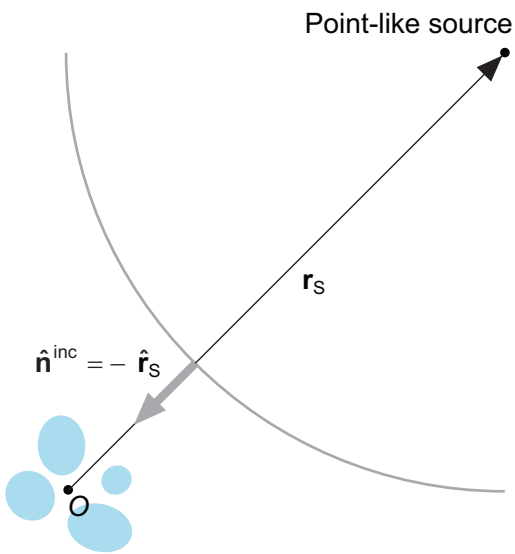

Fig. 2. Impressed point-like source located in the far zone of the scattering object.

valid in the limit $|k_1|r_S \to \infty$, $r_S/R_{INT} \to \infty$, and $r_S/(|k_1|R_{INT}^2) \to \infty$ (see Appendix B of Ref. [19]), where $r_S = |\mathbf{r}_S|$ is the distance from the origin to the source; $\hat{\mathbf{r}}_S = \mathbf{r}_S/r_S$ is the unit vector pointing from the origin to the source; and R_{INT} is the radius of the smallest circumscribing sphere of the object centered at O. Then for $\mathbf{r}'' \in V_{INT}$,

$$\mathbf{E}^S(\mathbf{r}'') = i\omega\mu_0 \exp\left(-ik_1\hat{\mathbf{r}}_S\cdot\mathbf{r}''\right) \times \frac{\exp(ik_1 r_S)}{4\pi r_S}\left(\overleftrightarrow{I} - \hat{\mathbf{r}}_S \otimes \hat{\mathbf{r}}_S\right)\cdot\mathbf{J}_0^S. \tag{62}$$

Obviously, $\mathbf{E}^S(\mathbf{r}'')$ is a transverse, homogeneous, *locally* plane electromagnetic wave

$$\mathbf{E}^S(\mathbf{r}'') = \mathbf{E}_0^{inc}\exp\left(ik_1\hat{\mathbf{n}}^{inc}\cdot\mathbf{r}''\right) \tag{63}$$

propagating in the direction

$$\hat{\mathbf{n}}^{inc} = -\hat{\mathbf{r}}_S \tag{64}$$

(see Fig. 2) and having the vector amplitude

$$\mathbf{E}_0^{inc} = i\omega\mu_0 \frac{\exp(ik_1 r_S)}{4\pi r_S}\left(\overleftrightarrow{I} - \hat{\mathbf{r}}_S \otimes \hat{\mathbf{r}}_S\right)\cdot\mathbf{J}_0^S \tag{65}$$

such that

$$\mathbf{E}_0^{inc}\cdot\hat{\mathbf{n}}^{inc} = 0. \tag{66}$$

Then the result of evaluating Eq. (26) for the impressed source (60) is

$$\mathbf{E}^{sca}(\mathbf{r}) = \int_{V_{INT}} d^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}') \cdot \int_{V_{INT}} d^3\mathbf{r}''\,\overleftrightarrow{T}(\mathbf{r}',\mathbf{r}'')\cdot\mathbf{E}_0^{inc}\exp\left(ik_1\hat{\mathbf{n}}^{inc}\cdot\mathbf{r}''\right). \tag{67}$$

This formula reproduces the result of the traditional approach to electromagnetic scattering based on the postulate that the homogeneous plane wave $\mathbf{E}^{inc}(\mathbf{r}) = \mathbf{E}_0^{inc}\exp(ik_1\hat{\mathbf{n}}^{inc}\cdot\mathbf{r})$ is the impressed incident field [1,5,10,19]. This demonstrates how a sufficiently distant point-like impressed source can effectively be "hidden" by replacing it with an impressed field in the form of a homogeneous plane electromagnetic wave propagating in a source-free space.

10. Far-field and general scattering dyadics

A well-known result of the conventional theory of electromagnetic scattering is the calculation of the scattered field generated by an impressed homogeneous plane electromagnetic wave in the far zone of the entire object (see, e.g., Refs. [1,5,10,19]). To recover this result, we use the assumptions and formulas of the preceding section and invoke the following asymptotic form of the first dyadic Green's function on the right-hand side of Eq. (67):

$$\overleftrightarrow{G}(\mathbf{r},\mathbf{r}') \to \left(\overleftrightarrow{I} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\frac{\exp(ik_1 r)}{4\pi r}\exp\left(-ik_1\hat{\mathbf{r}}\cdot\mathbf{r}'\right), \tag{68}$$

where $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/r$, and it is assumed that $|k_1|r \to \infty$, $r/R_{INT} \to \infty$, and $r/(|k_1|R_{INT}^2) \to \infty$. The result is the scattered field in the form of a transverse outgoing spherical wave given by

$$\mathbf{E}^{sca}(\mathbf{r}) \to \frac{\exp(ik_1 r)}{r}\overleftrightarrow{A}\left(\hat{\mathbf{r}},\hat{\mathbf{n}}^{inc}\right)\cdot\mathbf{E}_0^{inc}, \tag{69}$$

where

$$\overleftrightarrow{A}(\hat{\mathbf{r}},\hat{\mathbf{n}}^{inc}) = \frac{1}{4\pi}\left(\overleftrightarrow{I} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\right)\cdot\int_{V_{INT}} d^3\mathbf{r}'\exp\left(-ik_1\hat{\mathbf{r}}\cdot\mathbf{r}'\right) \times \int_{V_{INT}} d^3\mathbf{r}''\overleftrightarrow{T}(\mathbf{r}',\mathbf{r}'')\cdot\left(\overleftrightarrow{I} - \hat{\mathbf{n}}^{inc}\otimes\hat{\mathbf{n}}^{inc}\right) \times \exp\left(ik_1\hat{\mathbf{n}}^{inc}\cdot\mathbf{r}''\right) \tag{70}$$

is the so-called far-field scattering dyadic, while the radial unit vector $\hat{\mathbf{r}}$ plays the role of the scattering direction. It is easily seen indeed that

$$\mathbf{E}^{sca}(\mathbf{r})\cdot\hat{\mathbf{r}} = 0. \tag{71}$$

Note that the dyadic factor $\overleftrightarrow{I} - \hat{\mathbf{n}}^{inc}\otimes\hat{\mathbf{n}}^{inc}$ on the right-hand side of Eq. (70) does not follow directly from Eq. (67) and is included to make the far-field scattering dyadic reciprocal as a direct consequence of the symmetry relation (51) and Eq. (47):

$$\overleftrightarrow{A}\left(-\hat{\mathbf{n}}^{inc},-\hat{\mathbf{r}}\right) = \left[\overleftrightarrow{A}\left(\hat{\mathbf{r}},\hat{\mathbf{n}}^{inc}\right)\right]^T. \tag{72}$$

It is easily seen, however, that since the field (63) is a transverse plane electromagnetic wave, including this factor does not change Eq. (69) owing to Eq. (66): $(\overleftrightarrow{I} - \hat{\mathbf{n}}^{inc}\otimes\hat{\mathbf{n}}^{inc})\cdot\mathbf{E}_0^{inc} \equiv \mathbf{E}_0^{inc}$.

Note that in contrast to Eq. (51), the reciprocity relation (72) additionally includes the reversal of the incidence and scattering directions. The reciprocity of the far-field scattering dyadic has been well known in the discipline of electromagnetic scattering since its original (and somewhat less straightforward) derivation by Saxon [84]. In fact, it serves as the very origin of many reciprocity relations in the theory of radiative transfer and coherent backscattering [13]. Furthermore, it has routinely been used for testing the results of numerical simulations of electromagnetic scattering (see, e.g., Ref. [85]).

It is useful to generalize Eq. (69) by introducing the scattering dyadic $\overleftrightarrow{S}(\mathbf{r},\hat{\mathbf{n}}^{inc})$ operating throughout the entire space:

$$\mathbf{E}^{sca}(\mathbf{r}) = \overleftrightarrow{S}\left(\mathbf{r},\hat{\mathbf{n}}^{inc}\right)\cdot\mathbf{E}_0^{inc},\ \ \mathbf{r}\in\mathbb{R}^3. \tag{73}$$

Eq. (67) then implies

$$\overleftrightarrow{S}\left(\mathbf{r},\hat{\mathbf{n}}^{inc}\right) = \int_{V_{INT}} d^3\mathbf{r}'\,\overleftrightarrow{G}(\mathbf{r},\mathbf{r}')\cdot\int_{V_{INT}} d^3\mathbf{r}''\overleftrightarrow{T}(\mathbf{r}',\mathbf{r}'') \cdot\left(\overleftrightarrow{I} - \hat{\mathbf{n}}^{inc}\otimes\hat{\mathbf{n}}^{inc}\right)\exp\left(ik_1\hat{\mathbf{n}}^{inc}\cdot\mathbf{r}''\right). \tag{74}$$

Note again that the inclusion of the dyadic factor $\overleftrightarrow{I} - \hat{\mathbf{n}}^{inc}\otimes\hat{\mathbf{n}}^{inc}$ is not strictly necessary, but serves to make Eq. (74) asymptotically consistent with Eq. (69), as follows:

$$\overleftrightarrow{S}\left(r\hat{\mathbf{r}},\hat{\mathbf{n}}^{inc}\right) \underset{r\to\infty}{\to} \frac{\exp(ik_1 r)}{r}\overleftrightarrow{A}\left(\hat{\mathbf{r}},\hat{\mathbf{n}}^{inc}\right). \tag{75}$$

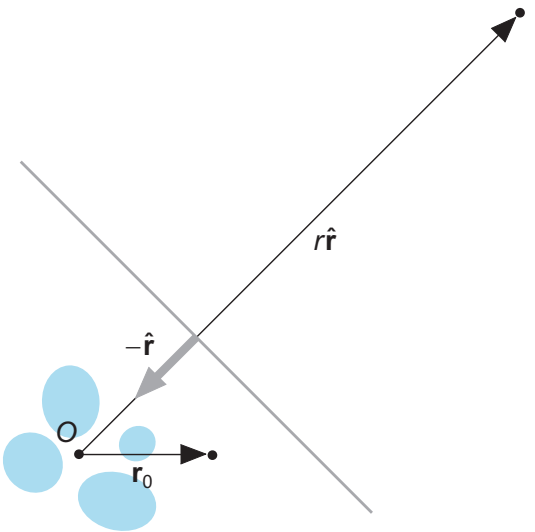

Fig. 3. Interpretation of Eq. (80).

11. Far-zone source Green's dyadic

Eq. (32) and the symmetry relation (59) imply a fundamental relationship between the general scattering dyadic $\overleftrightarrow{S}(\mathbf{r}, \hat{\mathbf{n}}^{\rm inc})$ introduced in the preceding section and the far-zone limit of the source Green's dyadic. Indeed, we first note that

$$\left[\overleftrightarrow{G}_{\rm S}(\mathbf{r}, \mathbf{r}_0)\right]^{\rm T} = \overleftrightarrow{G}_{\rm S}(\mathbf{r}_0, \mathbf{r}) = \overleftrightarrow{G}(\mathbf{r}_0, \mathbf{r}) + \int_{V_{\rm INT}} {\rm d}^3\mathbf{r}' \overleftrightarrow{G}(\mathbf{r}_0, \mathbf{r}') \cdot \int_{V_{\rm INT}} {\rm d}^3\mathbf{r}'' \overleftrightarrow{T}(\mathbf{r}', \mathbf{r}'') \cdot \overleftrightarrow{G}(\mathbf{r}'', \mathbf{r}). \tag{76}$$

Let us now assume that $|k_1|r \to \infty$, $r/R_{\rm INT} \to \infty$, and $r/(|k_1|R_{\rm INT}^2) \to \infty$, so that

$$\overleftrightarrow{G}(\mathbf{r}_0, \mathbf{r}) \to \left(\overleftrightarrow{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \frac{\exp({\rm i}k_1 r)}{4\pi r} \exp\left(-{\rm i}k_1 \hat{\mathbf{r}} \cdot \mathbf{r}_0\right) \tag{77}$$

and

$$\overleftrightarrow{G}(\mathbf{r}'', \mathbf{r}) \to \left(\overleftrightarrow{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \frac{\exp({\rm i}k_1 r)}{4\pi r} \exp\left(-{\rm i}k_1 \hat{\mathbf{r}} \cdot \mathbf{r}''\right). \tag{78}$$

Then comparison with Eq. (74) makes it obvious that

$$\lim_{r\to\infty} \left[\overleftrightarrow{G}_{\rm S}(\mathbf{r}, \mathbf{r}_0)\right]^{\rm T} = \frac{\exp({\rm i}k_1 r)}{4\pi r} \left[\exp\left(-{\rm i}k_1 \hat{\mathbf{r}} \cdot \mathbf{r}_0\right) \overleftrightarrow{I} + \overleftrightarrow{S}(\mathbf{r}_0, -\hat{\mathbf{r}})\right] \cdot \left(\overleftrightarrow{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \tag{79}$$

or

$$\lim_{r\to\infty} \overleftrightarrow{G}_{\rm S}(r\hat{\mathbf{r}}, \mathbf{r}_0) = \frac{\exp({\rm i}k_1 r)}{4\pi r} \left(\overleftrightarrow{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}\right) \cdot \left\{\exp\left(-{\rm i}k_1 \hat{\mathbf{r}} \cdot \mathbf{r}_0\right) \overleftrightarrow{I} + \left[\overleftrightarrow{S}(\mathbf{r}_0, -\hat{\mathbf{r}})\right]^{\rm T}\right\}. \tag{80}$$

In other words, the field generated by the object at a far-field observation point $r\hat{\mathbf{r}}$ in response to the point-like source centered at an arbitrary $\mathbf{r}_0 \in \mathbb{R}^3$ can be expressed in terms of the total field at $\mathbf{r}_0$ generated by the object in response to a far-field point-like source creating a homogeneous quasi-plane electromagnetic wave propagating in the direction $-\hat{\mathbf{r}}$ (see Fig. 3).

Eq. (80) generalizes Eq. (7.2.54) of Ref. [44] and can be considered an extension of Eq. (68). Interestingly, it can be used to calculate the energy radiated by a point-like source to infinity (as related to the radiative part of the enhancement of the decay rate of the point emitter near a nanoparticle [86]) using any standard computer solver capable of calculating the distribution of $\mathbf{E}(\mathbf{r})$ in the vicinity of the particle (the so-called near-field) under the plane-wave excitation [87].

Note also that the derivations in Sections 9–11 can be further streamlined by introducing additional operators. Proving their pseudo self-adjointness then immediately implies the corresponding reciprocity relations, as will be reported elsewhere.

12. Generalized Foldy equations

Let us now assume that $N \geq 2$ in Fig. 1 and generalize the famous Foldy equations [64] by making explicit use of the representation of the scattering object as a collection of non-overlapping distinct components. Let us re-write Eq. (34) as follows:

$$\overleftrightarrow{U}(\mathbf{r}, \mathbf{r}') = \sum_{i=1}^{N} \overleftrightarrow{U}_i(\mathbf{r}, \mathbf{r}'), \tag{81}$$

where

$$\overleftrightarrow{U}_i(\mathbf{r}, \mathbf{r}') \triangleq \begin{cases} \overleftrightarrow{0}, \ \mathbf{r} \notin V_{\rm INT}^i, \\ U(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')\overleftrightarrow{I}, \ \mathbf{r} \in V_{\rm INT}^i. \end{cases} \tag{82}$$

The next step is to introduce the ith-component transition dyadic $\overleftrightarrow{T}_i(\mathbf{r}, \mathbf{r}')$ with respect to the common coordinate system centered at O as the one satisfying the individual Lippmann–Schwinger equation formulated for the ith component of the object as if all the other components did not exist:

$$\hat{T}_i = \hat{U}_i + \hat{U}_i \hat{G} \hat{T}_i, \tag{83}$$

where we again use the operator notation introduced in Section 7. We complete the definition of the component transition dyadic by setting

$$\overleftrightarrow{T}_i(\mathbf{r}, \mathbf{r}') = \overleftrightarrow{0} \text{ unless } \mathbf{r} \in V_{\rm INT}^i \text{ and } \mathbf{r}' \in V_{\rm INT}^i \tag{84}$$

Let us further define the ith partial "exciting" field $\mathbf{E}_i(\mathbf{r})$ according to

$$E = \left(\hat{I} + \hat{G}\hat{T}_i\right) E_i \tag{85}$$

(cf. Eq. (40)), where the invertibility of $\hat{I} + \hat{G}\hat{T}_i$ is discussed later in this section. Eqs. (83) and (85) then imply

$$\hat{T}_i E_i = \hat{U}_i E. \tag{86}$$

In other words, E_i is the field that is transformed by the ith component transition dyadic into the polarization density inside this component (cf. Eq. (39)). Summing up all such partial exciting fields and using Eqs. (38) and (81) yields

$$E = E^{\rm S} + \sum_{i=1}^{N} \hat{G}\hat{T}_i E_i. \tag{87}$$

This, together with Eq. (85), leads to the following closed system of N integral equations for E_i:

$$E_i = E^{\rm S} + \sum_{j(\neq i)=1}^{N} \hat{G}\hat{T}_j E_j. \tag{88}$$

The electromagnetic Foldy equations (87) and (88) have been derived previously (see, e.g., Refs. [10,11,19,65]) but in a less straightforward way. It is easily seen that they imply the Neumann series

$$E = E^{\mathrm{S}} + \sum_{i=1}^{N} \hat{G}\hat{T}_i E^{\mathrm{S}} + \sum_{\substack{i=1\\ j(\neq i)=1}}^{N} \hat{G}\hat{T}_i\hat{G}\hat{T}_j E^{\mathrm{S}} + \sum_{\substack{i=1\\ j(\neq i)=1\\ l(\neq j)=1}}^{N} \hat{G}\hat{T}_i\hat{G}\hat{T}_j\hat{G}\hat{T}_l E^{\mathrm{S}} + \cdots, \tag{89}$$

which has often been cited as revealing "multiple scattering" by a multi-component object. It has been argued, however, that in the frequency domain, multiple scattering is a mathematical idealization rather than an actual physical phenomenon [88,89]. Another problem with the Neumann expansion is that it can be divergent [90,91].

Let us further rewrite Eqs. (41) and (83) as follows:

$$\hat{U} = \hat{T}\left(\hat{I} + \hat{G}\hat{T}\right)^{-1}, \tag{90}$$

$$\hat{U}_i = \hat{T}_i\left(\hat{I} + \hat{G}\hat{T}_i\right)^{-1}. \tag{91}$$

The invertibility of the operator $\hat{I} + \hat{G}\hat{T}$ follows from

$$\left(\hat{I} - \hat{G}\hat{U}\right)\left(\hat{I} + \hat{G}\hat{T}\right) = \hat{I}, \tag{92}$$

which is a direct consequence of Eq. (41), and the invertibility of $\hat{I} - \hat{G}\hat{U}$ discussed in Section 7, while the invertibility of $\hat{I} + \hat{G}\hat{T}_i$ is proven analogously.

Eqs. (81), (90), and (91) lead to a compact implicit relation between the total transition operator of the entire N-component object, $\hat{T}$, and the individual-component transition operators $\hat{T}_i$:

$$\hat{T}\left(\hat{I} + \hat{G}\hat{T}\right)^{-1} = \sum_{i=1}^{N} \hat{T}_i\left(\hat{I} + \hat{G}\hat{T}_i\right)^{-1}. \tag{93}$$

The explicit expression for $\hat{T}$ follows from Eqs. (41), (81), and (91):

$$\begin{aligned}\hat{T} &= \left(\hat{I} - \hat{U}\hat{G}\right)^{-1}\hat{U}\\ &= \left[\hat{I} - \sum_{i=1}^{N} \hat{T}_i\left(\hat{I} + \hat{G}\hat{T}_i\right)^{-1}\hat{G}\right]^{-1} \sum_{i=1}^{N} \hat{T}_i\left(\hat{I} + \hat{G}\hat{T}_i\right)^{-1}.\end{aligned} \tag{94}$$

Let us now define

$$\hat{X}_i \triangleq \left(\hat{I} + \hat{G}\hat{T}_i\right)^{-1}\left(\hat{I} + \hat{G}\hat{T}\right). \tag{95}$$

Together with Eq. (93), this yields

$$\hat{T} = \sum_{i=1}^{N} \hat{T}_i\hat{X}_i \tag{96}$$

and

$$\hat{X}_i = \hat{I} + \hat{G}\hat{T} - \hat{G}\hat{T}_i\hat{X}_i = \hat{I} + \hat{G}\sum_{j(\neq i)=1}^{N} \hat{T}_j\hat{X}_j, \tag{97}$$

which are completely analogous to Eqs. (87) and (88), since $E_i = \hat{X}_i E^{\mathrm{S}}$. Iterating Eqs. (96) and (97), we obtain the following representation:

$$\hat{T} = \sum_{i=1}^{N} \hat{T}_i + \sum_{\substack{i=1\\ j(\neq i)=1}}^{N} \hat{T}_i\hat{G}\hat{T}_j + \sum_{\substack{i=1\\ j(\neq i)=1\\ l(\neq j)=1}}^{N} \hat{T}_i\hat{G}\hat{T}_j\hat{G}\hat{T}_l + \cdots. \tag{98}$$

Of course, this series can also be derived as a direct corollary of Eqs. (40) and (89).

While being an analytically convenient and visual "multiple-scattering" representation, Eq. (98) is just the simplest iterative solution of Eq. (97). It does not necessarily converge, and even if it does, it is slower than more advanced numerical iterative methods, e.g., the conjugate-gradient technique, applied to the same equation. The latter has been previously discussed in the framework of the superposition T-matrix method and its numerical implementations such as that in Refs. [92–94]. Alternatively, one can solve Eq. (41) or the first part of Eq. (94) for $\hat{T}$ using a direct or iterative procedure, the benefit being that usually the individual transition operators $\hat{T}_i$ do not need to be known or evaluated separately. If the individual $\hat{T}_i$ are known then the computational cost of a single iteration will be comparable for both approaches. However, the iterative solution of Eq. (97) is expected to converge faster for well-separated moderately-sized (relative to the wavelength) particles, since most of the iterations in Eq. (41) will be spent on refining the details of each particle independently, which is equivalent to evaluating all $\hat{T}_i$.

13. Final remarks

Two implicit yet essential limitations of our discussion have so far been the following. First, we have refrained from the direct treatment of the singularity of the free-space dyadic Green's function $\overleftrightarrow{G}(\mathbf{r}, \mathbf{r}')$ at points $\mathbf{r}$ inside the cumulative scattering volume V_{INT} and/or inside the cumulative source volume V_{S}. Second, we have assumed that S_{INT} is sufficiently smooth and that the dielectric permittivity is a sufficiently smooth function of coordinates throughout each component volume V^i_{INT}.

These two particular aspects of the VIE formalism have thoroughly been analyzed in Ref. [73]. In view of that recent comprehensive study, one can safely assume that (i) all our formulas are valid in the sense of Eq. (21); and (ii) it is straightforward to further generalize all our results by considering the scattering object in the form of an arbitrary finite group of components made of nonmagnetic isotropic materials, including those with edges, corners, and intersecting internal interfaces.

The standard VIE formalism originally introduced by Saxon [95] has already been applied to a plethora of problems ranging from the development of the numerically exact "elastic" and "thermal" discrete dipole approximations (see, e.g., Refs. [53,81,96–98] and references therein) to the first-principles derivation of the "elastic" radiative transfer theory [13,19]. We hope that the extended formalism summarized in this tutorial brings a certain degree of self-consistency and closure and will help solve even more complex problems, including the first-principles derivation of the "thermal" radiative transfer theory.

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