TRANSFER OF POLARIZED RADIATION IN A MEDIUM CONSISTING OF FULLY ORIENTED STRONGLY ELONGATED PARTICLES.

GENERAL THEORY. "RAYLEIGH" SCATTERING

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The transfer of polarized radiation in a medium consisting of fully oriented strongly elongated particles (infinite cylinders) is studied. It is assumed that external linearly polarized radiation is incident perpendicular to the particle axes. Sobolev's method is used to find the radiation field in a semiinfinite homogeneous medium. Results of numerical calculations are given for very slender cylinders ("Rayleigh" scattering) and compared with corresponding data for an all-gas planetary atmosphere.

It was shown in [4] that the vector equation of radiation transfer in a medium consisting of fully oriented strongly elongated particles (infinite cylinders) can be split into two independent two-dimensional scalar transfer equations when external linearly polarized radiation is incident perpendicular to the direction of particle orientation. For a plane-parallel medium these equations have the form

$$\cos\theta \, \frac{dI_j(\tau_j, \theta, \theta_0)}{d\tau_j} = -I_j(\tau_j, \theta, \theta_0) + B_j(\tau_j, \theta, \theta_0), \quad j = l, r, \tag{1}$$

where the subscripts l and r correspond to the components polarized parallel and perpendicular to the particle axes. The boundaries of the medium are assumed to be parallel to the particle orientation direction. The angle $\theta \in [-\pi, \pi[$ characterizes the propagation direction of the scattered radiation. The external radiation is incident at an angle $\theta_0 \in [-\pi/2, \pi/2]$. The angles θ and θ_0 are measured from the inner normal to the boundary of the medium; clockwise readings correspond to positive values of the angle (Fig. 1, whose plane is perpendicular to the particle orientation direction).

The present paper is a continuation of [4]. It considers application of V. V. Sobolev's method [6, Chap. V] to solution of Eqs. (1) in the case of a semiinfinite homogeneous atmosphere. The case of "Rayleigh" scattering, which corresponds to the condition $d \ll \lambda$ (d is the particle diameter and λ is the wavelength of the light) is studied in detail and results of certain numerical calculations are reported.

References to the formulas in [4] are given in the form (4.n), where n is the formula number. Notation introduced there is generally not explained here.

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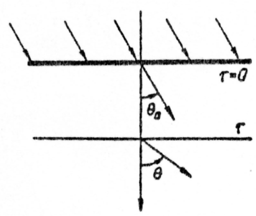

Fig. 1

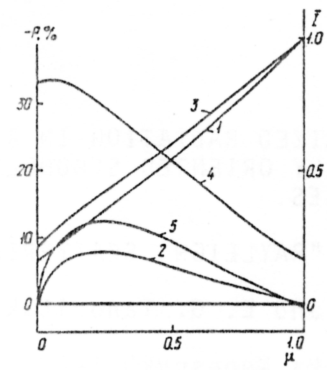

Fig. 2

Fig. 1. Coordinates in plane perpendicular to direction of particle orientation.

Fig. 2. Relative intensity (right-hand ordinate axis) and degree of polarization of radiation exiting at phase angle $\alpha = 0$ ($\mu = \mu_0$).

Basic relationships of the general theory for a semiinfinite atmosphere. In our problem, the source function B_j depends on two angle variables: $\theta \in [-\pi, \pi]$ and $\theta_0 \in [-\pi/2, \pi/2]$. We shall assume for simplicity that $\theta_0 \in [0, \pi/2]$, since it is obvious from symmetry considerations that $I_j(\tau_j, \theta, -\theta_0) = I_j(\tau_j, -\theta, \theta_0)$. Using Sobolev's method [6, Chap. V], we can express $B_j(\tau_j, \theta, \theta_0)$ in terms of a superposition of functions each of which depends only on one angle variable.

Omitting the subscript j for brevity and denoting $\mu_0 = \cos\theta_0$, $\mu_0 \in [0, 1]$, we write an initial relation for the source function:

$$B(\tau, \theta, \mu_0) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} I(\tau, \theta', \mu_0)\, \chi(\theta - \theta')\, d\theta' + \frac{\lambda}{2} S \chi(\theta - \theta_0) \exp(-\tau/\mu_0). \qquad (2)$$

Presenting formula (4.16) in the form

$$\chi[\cos(\theta - \theta_0)] = 1 + 2 \sum_{k=1}^{n} x_k [\cos k\theta \cos k\theta_0 + \sin k\theta \sin k\theta_0], \qquad (3)$$

we obtain instead of (2)

$$B(\tau, \theta, \mu_0) = B_0(\tau, \mu_0) + 2 \sum_{k=1}^{n} x_k [B_k(\tau, \mu_0) \cos k\theta + \tilde{B}_k(\tau, \mu_0) \sin k\theta], \qquad (4)$$

where

$$B_k(\tau, \mu_0) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} I(\tau, \theta', \mu_0) \cos k\theta' d\theta' + \frac{\lambda}{2} S \exp(-\tau/\mu_0) \cos k\theta_0, \qquad (5)$$

$$\tilde{B}_k(\tau, \mu_0) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} I(\tau, \theta', \mu_0) \sin k\theta' d\theta' + \frac{\lambda}{2} S \exp(-\tau/\mu_0) \sin k\theta_0. \qquad (6)$$

Denoting $\mu = \cos\theta$,

$$I_k(\tau, \mu, \mu_0) = \int_0^{\tau} B_k(\tau', \mu_0) \exp\left(-\frac{\tau - \tau'}{\mu}\right) \frac{d\tau'}{\mu}, \quad \mu > 0, \qquad (7)$$

$$I_k(\tau,\mu,\mu_0) = -\int_\tau^\infty B_k(\tau',\mu_0)\exp\left(-\frac{\tau-\tau'}{\mu}\right)\frac{d\tau'}{\mu}, \quad \mu<0 \tag{8}$$

(the $\tilde{I}_k$ are represented similarly), we obtain from (4)

$$I(\tau,\theta,\mu_0) = I_0(\tau,\mu,\mu_0) + 2\sum_{k=1}^{n} x_k\,[I_k(\tau,\mu,\mu_0)\cos k\theta + \tilde{I}_k(\tau,\mu,\mu_0)\sin k\theta]. \tag{9}$$

Rearranging in much the same way as in [6, Chap. V], we have

$$B_k(\tau,\mu_0) = R_{k0}(\mu_0)D(\tau,\mu_0) + H(\mu_0)\int_0^1 Q_k(\mu_0,\mu)D(\tau,\mu)\frac{d\mu}{\sqrt{1-\mu^2}}, \tag{10}$$

where $D(0,\mu) = \lambda SH(\mu)/2$,

$$Q_k(\mu_0,\mu) = \sum_{j=1}^{n}[q_j(\mu_0) + q_{j-2}(\mu_0)]\,g_{kj}(\mu), \tag{11}$$

and the function $D(\tau, \mu)$ is determined by the integral equation

$$D(\tau,\mu) = \int_0^\infty K(|t-\tau|)D(t,\mu)\,dt + \lambda S\exp(-\tau/\mu)/2, \tag{12}$$

whose kernel

$$K(\tau) = \int_0^1 \Psi_0(\mu)\exp(-\tau/\mu)\frac{d\mu}{\mu\sqrt{1-\mu^2}}. \tag{13}$$

The polynomials $R_{kj}(\mu)$ that we have introduced here are analogs of the corresponding Sobolev polynomials; they are determined from the recurrent relation

$$R_{k+1,j}(\mu) + R_{k-1,j}(\mu) = (2-\delta_{k0})\,\mu\,(1-\lambda x_k)R_{kj}(\mu) \tag{14}$$

and the initial conditions $R_{kk}(\mu) = 1$, $R_{kj}(\mu) = 0$ for $k<j$ $(j = 0, 1, 2, \ldots)$. Here δ_{kj} is the Kronecker delta. The expression for $g_{kj}(\mu)$ has the form

$$g_{kj}(\mu) = \Psi_0(\mu)R_{kj}(\mu) - \Psi_j(\mu)R_{k0}(\mu), \tag{15}$$

where

$$\Psi_j(\mu) = \frac{\lambda}{\pi}\left[\delta_{j0} + 2\sum_{i=1}^{n} x_i R_{ij}(\mu)T_i(\mu)\right]. \tag{16}$$

and the $T_j(\mu) = \cos[j\arccos\mu]$ are Chebyshev polynomials of the first kind. The characteristic function $\Psi_0(\mu)$ that appears in (13) is a polynomial of even degree in μ. As for the $q_j(\mu_0)$, they are polynomials of degree n in μ_0 and can be determined from the system of linear algebraic equations

$$q_j(\mu_0) = R_{j0}(\mu_0) + \sum_{i=1}^{n}[q_i(\mu_0) + q_{i-2}(\mu_0)]\int_0^1 \frac{g_{ji}(\mu)H(\mu)\,d\mu}{\sqrt{1-\mu^2}}. \tag{17}$$

Here $H(\mu)$ is assumed to be known.

The formulas used to find the $\tilde{B}_k(\tau,\mu_0)$ $(k = 1, 2, 3, \ldots)$ are similar to those given above:

$$\tilde{B}_k(\tau,\mu_0) = \Big[R_{k1}(\mu_0)\tilde{D}(\tau,\mu_0) + + \tilde{H}(\mu_0)\int_0^1 \tilde{Q}_k(\mu_0,\mu)\tilde{D}(\tau,\mu)\sqrt{1-\mu^2}d\mu\Big]\sqrt{1-\mu_0^2}, \tag{18}$$

where $\tilde{D}(0,\mu) = \lambda S H(\mu)/2$,

$$\tilde{Q}_k(\mu_0,\mu) = \sum_{j=2}^{n} [\tilde{q}_j(\mu_0) + \tilde{q}_{j-2}(\mu_0)]\ \tilde{g}_{kj}(\mu), \tag{19}$$

$$\tilde{g}_{kj}(\mu) = \tilde{\Psi}_1(\mu) R_{kj}(\mu) - \tilde{\Psi}_j(\mu) R_{k1}(\mu), \tag{20}$$

$$\tilde{\Psi}_j(\mu) = \frac{2\lambda}{\pi}\sum_{i=1}^{n} x_i U_{i-1}(\mu) R_{ij}(\mu), \tag{21}$$

and the $U_{j-1}(\mu) = \sin[j\arccos\mu]/\sqrt{1-\mu^2}$ are Chebyshev polynomials of the second kind. The function $\tilde{D}(\tau, \mu)$ is determined by Eq. (12), in which

$$K(\tau) \equiv \tilde{K}(\tau) = \int_0^1 \tilde{\Psi}_1(\mu)(1-\mu^2)\exp(-\tau/\mu)\frac{d\mu}{\mu\sqrt{1-\mu^2}}, \tag{22}$$

where $\tilde{\Psi}_1(\mu)(1-\mu^2)$ is also an even polynomial in μ and of the same degree as $\Psi_0(\mu)$. As a result, the theory of finding the $\tilde{B}_k(\tau, \mu_0)$ is formally no different from the theory derived for the $B_k(\tau, \mu_0)$. Finally, the $\tilde{q}_j(\mu_0)$ are polynomials of degree $n - 1$ in μ_0 and can be found from the system of linear algebraic equations

$$\tilde{q}_j(\mu_0) = R_{j1}(\mu_0) + \sum_{i=2}^{n}[\tilde{q}_i(\mu_0) + \tilde{q}_{i-2}(\mu_0)]\int_0^1 \tilde{g}_{ji}(\mu)\tilde{H}(\mu)\sqrt{1-\mu^2}d\mu. \tag{23}$$

Having determined the functions

$$\varphi_k(\mu) = \frac{2}{\lambda S}B_k(0,\mu), \quad \tilde{\varphi}_k(\mu) = \frac{2}{\lambda S}\tilde{B}_k(0,\mu), \tag{24}$$

it is easy to obtain the following expression for the reflectance of a semi-infinite atmosphere:

$$\rho(\theta,\theta_0) = \frac{\lambda}{2}\frac{1}{\mu+\mu_0}\Big\{\varphi_0(\mu)\varphi_0(\mu_0) + + 2\sum_{k=1}^{n}(-1)^k x_k[\varphi_k(\mu)\varphi_k(\mu_0) \mp \tilde{\varphi}_k(\mu)\tilde{\varphi}_k(\mu_0)]\Big\}. \tag{25}$$

The minus sign is taken in (25) for $\theta\in[0,\pi/2]$, and the plus sign for $\theta\in[-\pi/2,0]$. The functions $\phi_k(\mu)$ and $\tilde{\phi}_k(\mu)$ can be found from the formulas

$$\varphi_k(\mu) = q_k(\mu)H(\mu), \quad \tilde{\varphi}_k(\mu) = \tilde{q}_k(\mu)\tilde{H}(\mu)\sqrt{1-\mu^2}. \tag{26}$$

In turn, the corresponding analogs of the Ambartsumyan-Chandrasekhar equations [6, Chap. V, § 3] can be derived for $H(\mu)$ and $\tilde{H}(\mu)$. That is to say, $H(\mu)$ satisfies the integral equation

$$H(\mu) = 1 + \mu H(\mu) \int_0^1 \frac{\Psi_0(\mu') H(\mu') \, d\mu'}{(\mu + \mu') \sqrt{1 - \mu'^2}}. \tag{27}$$

Since this implies

$$\int_0^1 \frac{H(\mu) \Psi_0(\mu) \, d\mu}{\sqrt{1 - \mu^2}} = 1 - \left[1 - 2 \int_0^1 \frac{\Psi_0(\mu) \, d\mu}{\sqrt{1 - \mu^2}}\right]^{1/2}, \tag{28}$$

we can write Eq. (27) in the alternative form

$$\frac{1}{H(\mu)} = \left[1 - 2 \int_0^1 \frac{\Psi_0(\mu') \, d\mu'}{\sqrt{1 - \mu'^2}}\right]^{1/2} + \int_0^1 \frac{H(\mu') \Psi_0(\mu') \mu' d\mu'}{(\mu + \mu') \sqrt{1 - \mu'^2}}. \tag{29}$$

There is also a linear singular equation with a Cauchy kernel:

$$H(\mu) T(\mu) = 1 + \mu \int_0^1 \frac{\Psi_0(\mu') H(\mu') \, d\mu'}{(\mu' - \mu) \sqrt{1 - \mu'^2}}, \tag{30}$$

where

$$T(\mu) = 1 + \mu \int_{-1}^{+1} \frac{\Psi_0(\mu') \, d\mu'}{(\mu' - \mu) \sqrt{1 - \mu'^2}}. \tag{31}$$

We note that in the case at hand, in contrast to the three-dimensional medium, the function $T(\mu)$ for $\mu \in [0, 1)$ is a polynomial in μ of the same degree as the polynomial $\Psi_0(\mu)$ since for $\mu \in [0, 1)$ $\int_{-1}^{+1} [(\mu - \mu')(1 - \mu'^2)^{1/2}]^{-1} d\mu' \equiv 0.$

The equations for the function $\tilde{H}(\mu)$ differ in no respect from those given above. In those equations, as we noted, it is sufficient to replace the polynomial $\Psi_0(\mu)$ with the polynomial $\tilde{\Psi}_1(\mu)(1 - \mu^2)$, in which the function $\Psi_1(\mu)$ is determined by expression (21).

Thus, determination of the source functions $B_{l,r}(\tau_{l,r}, \theta, \mu_0)$ generally reduces to solution of only four integral equations of the type (12) separately for the functions $D_{l,r}(\tau_{l,r}, \mu)$ and $\tilde{D}_{l,r}(\tau_{l,r}, \mu)$.

Furthermore, as in the three-dimensional scalar case [10], the functions $I_k(\tau, \mu, \mu_0)$ and $\tilde{I}_k(\tau, \mu, \mu_0)$ can be expressed in terms of the corresponding reduced radiation intensities $J(\tau, \mu, \mu_0)$ and $\tilde{J}(\tau, \mu, \mu_0)$. These functions can be expressed directly in terms of $D(\tau, \mu)$ and $\tilde{D}(\tau, \mu)$ which are specified only at depth τ and on the upper boundary of the medium, i.e., no integration over a space coordinate is necessary.

Thus, the only formal difference between the theory considered here and the scalar three-dimensional theory of light scattering in planetary atmospheres [6] is that the basic integral equation (12) has a kernel of the form (13) in the former case and one of the form $K(\tau) = \int_0^1 \Psi(\mu) \exp(-\tau/\mu) \, d\mu/\mu$ in the latter, i.e., it does not have $\sqrt{1 - \mu^2}$ under the integral sign in the denominator.

"Rayleigh" scattering. Let us apply the formulas derived above in the simplest case of "Rayleigh" scattering, in which it is assumed that the diameter of the cylindrical particles is much smaller than the wavelength of the light. As we noted above, the present three-dimensional vector problem of radiation transfer breaks up into two independent two-dimensional scalar problems, and the corresponding scattering phase functions in our case will be [1, Chap. 8; 2, Chap. 15] $\chi_l(\theta) \equiv 1$, $\chi_r(\theta) = 1 + \cos 2\theta$. Let us consider these problems separately.

1. Scattering phase function $\chi_l(\theta) = 1$. Omitting the subscript l for simplicity, we have $\Psi_0(\mu) = \lambda/\pi$, $B(\tau, \theta, \mu_0) = B_0(\tau, \mu_0) = D(\tau, \mu_0)$,

$$D(\tau, \mu_0) = \frac{\lambda}{\pi} \int_0^\infty K_0(|t - \tau|) D(t, \mu_0)\, dt + \lambda S \exp(-\tau/\mu_0)/2, \tag{32}$$

where

$$K_0(\tau) = \int_0^1 \frac{\exp(-\tau/\mu)\, d\mu}{\mu \sqrt{1 - \mu^2}} = \int_1^\infty \frac{\exp(-\tau x)\, dx}{\sqrt{x^2 - 1}} \tag{33}$$

is a Macdonald function.

It is interesting to note that Eq. (32) with kernel (33) is the electromagnetic-wave shore-refraction equation discussed in detail in [3,7]. Unfortunately, it is not yet clear why the two physically different problems reduce to a single equation.

We obtain for the reflectance

$$\rho(\theta, \theta_0) = \lambda H(\mu) H(\mu_0)/[2(\mu + \mu_0)], \tag{34}$$

and the corresponding analog of the Ambartsumyan equation for the H function has the form

$$\varphi_0(\mu) = H(\mu) = 1 + \frac{\lambda}{\pi} \mu H(\mu) \int_0^1 \frac{H(\mu')\, d\mu'}{(\mu + \mu') \sqrt{1 - \mu'^2}}. \tag{35}$$

Equation (29) with the characteristic function $\Psi_0(\mu) = \lambda/\pi$ was used to find numerical values of $H(\mu)$. The equation was solved by an iterative procedure. The technique proposed in [11] was used to accelerate convergence for conservative or nearly conservative scattering. Calculated results for certain λ_l appear in Table 1.

In our case the function $T(\mu) = 1$ for $\mu \in [0, 1)$. Therefore the method described in [5, p. 150; 9] can be used to obtain the following explicit expression for $H(\mu)$:

$$H(\mu) = \left[\frac{(1 - k\mu)(1 + \mu)}{(1 + \lambda)(1 + k\mu)}\right]^{1/2} \exp[-\omega(\mu)], \tag{36}$$

where $k^2 = 1 - \lambda^2$

$$\omega(\mu) = \frac{\lambda}{\pi} \int_0^1 \frac{\ln(|\eta - \mu|)\, d\eta}{(1 - k^2\eta^2) \sqrt{1 - \eta^2}}.$$

In the conservative case ($\lambda = 1$), formula (36) can be reduced to the form

Table 1

Values of the Function $H_l(\mu)$

μ	λ_l							
	1.000	0.995	0.975	0.950	0.900	0.800	0.700	0.500
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.1912	1.1791	1.1634	1.1512	1.1336	1.1082	1.0885	1.0570
0.2	1.3530	1.3259	1.2922	1.2671	1.2320	1.1835	1.1475	1.0926
0.3	1.5074	1.4627	1.4090	1.3700	1.3169	1.2461	1.1951	1.1202
0.4	1.6582	1.5934	1.5176	1.4640	1.3926	1.3001	1.2354	1.1428
0.5	1.8067	1.7195	1.6200	1.5511	1.4613	1.3478	1.2703	1.1619
0.6	1.9537	1.8418	1.7171	1.6324	1.5241	1.3904	1.3009	1.1783
0.7	2.0997	1.9609	1.8095	1.7088	1.5820	1.4288	1.3281	1.1925
0.8	2.2449	2.0770	1.8979	1.7807	1.6356	1.4636	1.3525	1.2051
0.9	2.3895	2.1905	1.9825	1.8487	1.6855	1.4954	1.3745	1.2162
1.0	2.5337	2.3016	2.0636	1.9132	1.7321	1.5246	1.3945	1.2262

$$H^0(\cos\theta) = (1+\cos\theta)^{1/2} \exp\{[\mathrm{Cl}_2(\pi/2+\theta) + \mathrm{Cl}_2(\pi/2-\theta)]/\pi\}, \qquad (37)$$

where $\mathrm{Cl}_2(x) = -\int\limits_0^x \ln[2\sin(y/2)]\,dy$ is the Clausen integral. In particular, for $\theta = 0$ we have $H^0(1) = \sqrt{2}\exp(2G/\pi) = 2.5337\ldots$, where $G = \mathrm{Cl}_2(\pi/2) = 0.915\,97\ldots$ is Catalan constant.

2. Scattering phase function $\chi_r(\theta) = 1 + \cos 2\theta$. In this case $x_1 = 0$ and $x_2 = 1/2$, and we have instead of (25) (the subscript r is omitted for simplicity):

$$\rho(\theta, \theta_0) = \lambda\,[\varphi_0(\mu)\,\varphi_0(\mu_0) + \varphi_2(\mu)\,\varphi_2(\mu_0) \mp \tilde{\varphi}_2(\mu)\,\tilde{\varphi}_2(\mu_0)]/[2(\mu+\mu_0)], \qquad (38)$$

where the characteristic functions

$$\Psi_0(\mu) = 2\lambda\,[1 - (2-\lambda)\,\mu^2 + 2\,(1-\lambda)\,\mu^4]/\pi, \qquad (39)$$

$$\tilde{\Psi}_1(\mu) = 4\lambda\mu^2/\pi. \qquad (40)$$

For the ϕ functions we can derive the expressons

$$\varphi_0(\mu) = [1 + q(\mu)\,(\alpha_0 - 2\alpha_2)/2 + q_1(\mu)\,(\alpha_1 - 2\alpha_3)]\,H(\mu), \qquad (41)$$

$$\varphi_2(\mu) = [2\,(1-\lambda)\,\mu^2 - 1 + \alpha_0 q(\mu)/2 + \alpha_1 q_1(\mu)]\,H(\mu), \qquad (42)$$

where $q(\mu) = q_0(\mu) + q_2(\mu)$,

$$q(\mu) = \frac{2(1-\lambda)\,[\alpha_1 - \alpha_3 + (1 - \alpha_0 + \alpha_2)\,\mu]\,\mu}{(1-\alpha_0+\alpha_2)^2 + (1-\lambda)\,(\alpha_1-\alpha_3)\,(2\alpha_3-\alpha_1)}, \qquad (43)$$

$$q_1(\mu) = \frac{(1-\lambda)\,[1 - \alpha_0 + \alpha_2 - (1-\lambda)\,(2\alpha_3-\alpha_1)\,\mu]}{(1-\alpha_0+\alpha_2)^2 + (1-\lambda)\,(\alpha_1-\alpha_3)\,(2\alpha_3-\alpha_1)}, \qquad (44)$$

and $\alpha_n\ (n = 0, 1, 2, \ldots)$ denote the corresponding moments of the H functions:

$$\alpha_n = \frac{2\lambda}{\pi}\int\limits_0^1 \frac{H(\mu)\,\mu^n d\mu}{\sqrt{1-\mu^2}}. \qquad (45)$$

The relation $\alpha_0 - (2-\lambda)\,\alpha_2 + 2\,(1-\lambda)\,\alpha_4 = 1 - [(1-\lambda)\,(2-\lambda)/2]^{1/2}$, which proceeds from (28) and (39), also holds.

Table 2

Values of the Function $H_r(\mu)$ and Its Moments

μ	λ_r							
	1.000	0.995	0.975	0.950	0.900	0.800	0.700	0.500
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.2900	1.2761	1.2562	1.2396	1.2142	1.1755	1.1443	1.0934
0.2	1.5255	1.4935	1.4502	1.4158	1.3653	1.2920	1.2357	1.1484
0.3	1.7480	1.6940	1.6241	1.5705	1.4941	1.3875	1.3085	1.1903
0.4	1.9639	1.8845	1.7851	1.7110	1.6082	1.4695	1.3696	1.2244
0.5	2.1760	2.0676	1.9362	1.8407	1.7113	1.5414	1.4222	1.2530
0.6	2.3856	2.2450	2.0792	1.9616	1.8054	1.6055	1.4683	1.2774
0.7	2.5933	2.4173	2.2152	2.0748	1.8921	1.6633	1.5092	1.2987
0.8	2.7998	2.5853	2.3449	2.1814	1.9722	1.7156	1.5459	1.3175
0.9	3.0052	2.7492	2.4691	2.2821	2.0467	1.7634	1.5789	1.3341
1.0	3.2099	2.9095	2.5881	2.3774	2.1163	1.8072	1.6089	1.3490
α_0	2.4492	2.2693	2.0396	1.8648	1.6174	1.2734	1.0180	0.6335
α_1	1.7607	1.6148	1.4338	1.2996	1.1138	0.8627	0.6815	0.4163
α_2	1.4492	1.3231	1.1687	1.0554	0.9001	0.6928	0.5448	0.3306
α_3	1.2607	1.1480	1.0112	0.9112	0.7752	0.5946	0.4665	0.2821

In the conservative case ($\lambda = 1$) we obtain instead of (41) and (42), respectively.

$$\varphi_0^0(\mu) = [1 + (2 - \alpha_0^0)\,\mu/\alpha_1^0]\,H^0(\mu); \qquad \varphi_2^0(\mu) = (\alpha_0^0\mu/\alpha_1^0 - 1)\,H^0(\mu), \tag{46}$$

where the superscript 0 is used to identify the values when $\lambda = 1$. It can be shown that $\alpha_0^0 - \alpha_2^0 = 1$, $\alpha_1^0 - \alpha_3^0 = 1/2$, $(\alpha_1^0)^2 - (\alpha_2^0)^2 = 1$.

Finally,

$$\tilde{\varphi}_2(\mu) = 2\mu\tilde{H}(\mu)\sqrt{1-\mu^2}. \tag{47}$$

Values of the functions $H_r(\mu)$ and $\tilde{H}_r(\mu)$ and the corresponding moments are given in Tables 2 and 3. They were obtained by numerical solution of Eq. (29), in which we substituted the characteristic function (39) and the polynomial $\tilde{\Psi}_1(\mu)(1-\mu^2)$, respectively, with $\tilde{\Psi}_1(\mu)$ given by expression (40). Further, the following values of the diffusion coefficient k_r were calculated from Eq. (4.21):

λ_r	1.000	0.995	0.975	0.950	0.900	0.800	0.700	0.500
k_r	0.0000	0.0998	0.2209	0.3086	0.4265	0.5781	0.6818	0.8250

They may be needed for use with the asymptotic formulas given in [4].

3. Milne problem for conservative scattering. Using formulas (4.35), (34) and (35), it is easily found that $u_l^0(\arccos\mu) = H_l^0(\mu)/\sqrt{2}$. Similarly, we find $u_r^0(\arccos\mu) = [\varphi_{r0}^0(\mu) + \varphi_{r2}^0(\mu)]/(2\mu) = H_r^0(\mu)/\alpha_{r1}^0$ from (4.35), (38) and (46).

Thus, we now have all of the formulas and tables necessary to find the Stokes parameters I_l and I_r of radiation exiting a semiinfinite homogeneous medium after "Rayleigh" scattering within the frame work of the present vector transfer problem.

Comparison of characteristics of radiation exiting isotropic and anisotropic Rayleigh atmospheres. As we know, many scattering media (and planetary atmospheres in particular) contain considerable numbers of partially or fully

Table 3

Values of the Function $\tilde{H}_r(\mu)$

μ	λ_r							
	1.000	0.995	0.975	0.950	0.900	0.800	0.700	0.500
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0425	1.0422	1.0412	1.0399	1.0374	1.0325	1.0278	1.0191
0.2	1.0760	1.0755	1.0736	1.0712	1.0665	1.0576	1.0492	1.0335
0.3	1.1035	1.1028	1.1001	1.0967	1.0903	1.0779	1.0663	1.0450
0.4	1.1266	1.1257	1.1223	1.1182	1.1101	1.0948	1.0805	1.0544
0.5	1.1464	1.1453	1.1414	1.1365	1.1270	1.1091	1.0925	1.0622
0.6	1.1635	1.1624	1.1579	1.1523	1.1416	1.1214	1.1027	1.0689
0.7	1.1785	1.1773	1.1723	1.1662	1.1544	1.1321	1.1116	1.0747
0.8	1.1918	1.1905	1.1850	1.1784	1.1656	1.1415	1.1194	1.0797
0.9	1.2037	1.2022	1.1964	1.1893	1.1756	1.1499	1.1263	1.0841
1.0	1.2143	1.2127	1.2066	1.1990	1.1845	1.1573	1.1324	1.0880

oriented nonspherical particles. Calculation of radiation fields is an extremely difficult problem in such media. Accordingly, there is definite interest in comparison of the intensities and degrees of polarization on the basis of exact calculations in two extreme particular cases: in an isotropic medium with Rayleigh scattering and in an anisotropic medium that consists of fully oriented slender ("Rayleigh") cylinders. This comparison will permit rigorous evaluation of the degree to which the anisotropy of the medium influences the characteristics of the scattered radiation.

Because of the differences in the scattering geometries in isotropic and anisotropic media, it is possible to compare only dimensionless characteristics of the radiation - its relative intensity (brightness distribution) and degree of polarization. In fact, light incident along the normal to the axis of an infinite cylinder is scattered only in the plane perpendicular to the axis and not in all directions, as in the case of a particle of finite size. Therefore the very dimensions of the Stokes parameters change: J/(cm^2·sec·rad) in the anisotropic case instead of J(cm^2·sec·sr) in the isotropic case.

As we know [8, Chap. I], the intensity I and degree of polarization P of radiation are found from the formulas

$$I = I_{\perp} + I_{\parallel}, \quad P = (I_{\perp} - I_{\parallel})/I, \qquad (48)$$

where the subscripts $\perp$ and $\parallel$ pertain to the radiations polarized perpendicular and parallel, respectively, to the plane that contains the light-propagation direction and the normal to the boundary of the medium (the meridional plane). In our anisotropic medium, the meridional plane is perpendicular to the particle axes. Therefore formulas (48) assume the form $I_a = I_{la} + I_{ra}$, $P_a = (I_{la} - I_{ra})/I_a$. Here and below, the subscripts a and i will identify radiation characteristics in anisotropi and isotropic media, respectively.

Let us compare the characteristics of radiation exiting a semiinfinite homogeneous conservatively scattering medium. The radiation incident on the medium is assumed to be unpolarized. In the case of the anisotropic medium, the radiation falls perpendicular to the particle orientation direction. $I_a(0, \theta, \theta_0)$ and $P_a(0, \theta, \theta_0)$ are calculated from the formulas and tables given above and $I_i(0, \mu, \mu_0, \phi)$ and $P_i(0, \mu, \mu_0, \varphi)$ from the formulas and tables of [8, Chap. X].

Curves 1 and 2 in Fig. 2 represent $\bar{I}_i(\mu) = I_i(0, -\mu, \mu, \pi)/I_i(0, -1, 1, \pi)$ and $P_i(0, -\mu,$

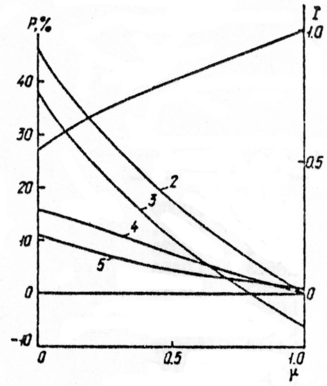

Fig. 3. The same as Fig. 2 for $\mu_0 = 0$.

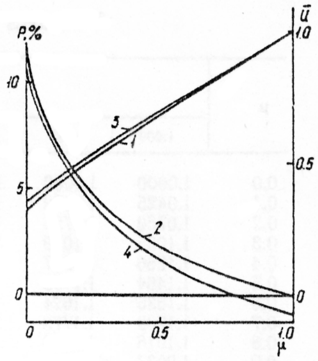

Fig. 4. The same as Fig. 2 in Milne problem.

$\mu, \pi)$, $\mu \in [0, 1]$, which describe the brightness distribution and degree of polarization of the reflected radiation along the intensity equator of a planet with an all-gas atmosphere at phase angle $\alpha = 0$. Curves 3 and 4 represent the anisotropic analogs of these quantities, $\bar{I}_a(\mu) = I_a(0, \pi - \arccos\mu, -\arccos\mu)/I_a(0, \pi, 0)$ and $P_a(0, \pi - \arccos\mu, -\arccos\mu)$. We see that the differences between the brightness distribution curves are comparatively minor. At the same time, the polarization curves differ strongly, and the difference becomes fundamental at $\mu = 0$ and $\mu = 1$. This is because only first-order scattering, which is unpolarized in the isotropic case and has a 33.3(3)% degree of polarization in the anisotropic case, contributes to the reflected radiation when $\mu = 0$. When $\mu = 1$, the isotropic medium possesses axial symmetry, so that the emerging radiation is unpolarized. The anisotropic medium does not possess such symmetry when $\mu = 1$. Therefore the degree of polarization of the exiting radiation is nonzero.

For comparison, Fig. 2 includes a curve of $P'_a = [I_{la} - I_{ra} - (I^{(1)}_{la} - I^{(1)}_{ra})]/I_a$ (curve 5), where $I^{(1)}_{la}$ and $I^{(1)}_{ra}$ describe the contribution of first-order scattering. The curve of the analogous P'_i coincides with curve 2 in this case. P' is equal to the degree of polarization of the reflected radiation if the singly scattered exiting radiation is unpolarized. Qualitatively, curve 2 and 5 behave practically identically. In particular, P'_a is near zero at the point $\mu = 1$. This indicates that photons that have been scattered two or more times have little "memory" of the anisotropy of the medium.

Figure 3 shows curves of the same quantities, but for the case in which the light is incident along the normal to the boundary of the medium ($\mu_0 = 1$). The relative intensities $\bar{I}_i(\mu) = I_i(0, -\mu, 1, 0)/I_i(0, -1, 1, 0)$ and $\bar{I}_a(\mu) = I_a(0, \pi - \arccos\mu, 0)/I_a(0, \pi, 0)$ were equal at the accuracy with which the curves could be plotted (curve 1). The degrees of polarization $P_i(0, -\mu, 1, 0)$ (curve 2) and $P_a(0, \pi - \arccos\mu, 0)$ (curve 3) differ insignificantly, but the difference again becomes fundamental for $\mu = 1$. Curves 4 and 5 represent $P_i'(0, -\mu, 1, 0)$ and $P_a'(0, \pi - \arccos\mu, 0)$, and do not include the contribution of first-order scattering to the polarization of the escaping radiation. Qualitatively, these quantities behave practically identically (in particular, near $\mu = 1$), which again confirms out inference that the anisotropy of the medium is a decisive factor in the polarization of only the singly scattered light.

Finally, let us consider the Milne problem, numerical results for which appear in Fig. 4. In this case, all of the photons exiting the medium have been multiply scattered. We see that both the relative intensities $\bar{u}_i(\mu)=u_i(\mu)/u_i(1)$ (curve 1) and $\bar{u}_a(\mu)=\bar{u}_a(\arccos\mu)/u_a(0)$ (curve 3) and the degrees of polarization $P_i(\mu)$ (curve 2) and $P_a(\arccos\mu)$ (curve 4) practically coincided. Although P_α does not vanish at $\mu = 1$ (axial symmetry is absent), it is still quite small.

Thus, we may draw the following conclusions as to the influence of anisotropy on the characteristics of the escaping radiation in the case of conservative Rayleigh scattering from the numerical results presented above: 1) anisotropy of the medium has a strong influence on the degree of polarization of only the singly scattered radiation, and the qualitative behavior of the degree of polarization of light scattered two or more times in an anisotropic medium is found to be practically the same as in the isotropic case; 2) anisotropy of the medium has practically no influence on the relative angular dependence of exiting-radiation intensity.

Needless to say, these conclusions require further careful study in cases in which the light is scattered on coarse absorbing and nonabsorbing particles. It would be expected that the difference between the isotropic and anisotropic media would be much smaller in this case (see, for example, [1, Chap. 8]). In any event, it should be noted that the estimates given above for the influence of anisotropy on escaping-radiation characteristics in the Rayleigh case are based on rigorous calculations and, to the best of our knowledge, have not been made previously.

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