

THE TRANSFER OF POLARIZED RADIATION IN A MEDIUM CONSISTING OF COMPLETELY ORIENTED HIGHLY ELONGATED PARTICLES

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In this paper, we discuss the transfer of polarized radiation in a homogeneous, plane-parallel medium consisting of completely oriented highly elongated particles (infinite cylinders) when linearly polarized radiation is incident perpendicular to the orientation of the particles. It is shown that the scattering problem is geometrically two-dimensional; the vector equation of transfer splits into two independent two-dimensional scalar equations of transfer. The main equations and formulas are given for the case of a semi-infinite medium.

INTRODUCTION

The study of the optical properties of anisotropic media containing partially or completely oriented aspherical particles is an important part of many problems in astrophysics and atmospheric optics. Since it is very complicated to calculate the characteristics of electromagnetic radiation scattered by aspherical particles of arbitrary shape and orientation, there is some value in approximating these particles by the simplest aspherical particle for which the scattering problem has been rigorously solved - an infinite cylinder of circular cross section [3,7]. Of course, this is a quite crude approximation; however, highly elongated particles (for which this approximation yields good results) do occur in nature. Examples include the ice needles in terrestrial ice clouds; these needles range up to 3 mm in length, with a diameter of up to 150 μm [3]. The infinite cylinder approximation is frequently used in studying the optical properties of interstellar dust grains, which are generally aspherical and partially oriented, as indicated by the existence of interstellar light polarization [6,9].

It is well known [7] that when a plane electromagnetic wave is incident on an infinite circular cylinder at an angle α to the axis of the cylinder, the scattered radiation lies on the surface of a cone with vertex angle 2α . The axis of this cone coincides with the axis of the cylinder. If a cylinder of the same diameter has a finite length L , the scattering results in a divergent spherical wave; the amplitude functions for a finite cylinder with $L \gg \lambda$ and $L \gg d$ (λ is the wavelength of the light, and d is the diameter of the cylinder) only differ from the analogous characteristics for light scattered by an infinite cylinder by an additional factor of [3]

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$$R(L) = \frac{\sin [(\pi L/\lambda) (\cos \varphi - \cos \alpha)]}{(\pi L/\tilde{\lambda}) (\cos \varphi - \cos \alpha)} \cdot \frac{2\pi L}{\tilde{\lambda}}, \quad (1)$$

due to diffraction at the ends of the cylinder. This factor has an asymptotic $R(L) \xrightarrow{L \rightarrow \infty} 2\pi\delta(\cos \varphi - \cos \alpha)$, where φ is the angle between the scattered light and the cylinder axis, and $\delta(x)$ is the Dirac delta function.

Thus, the intensity of the scattered radiation is highest for directions lying on the surface of the infinite-cylinder scattering cone. This maximum turns out to be sharper for longer cylinders. This simple relationship in fact allows us to estimate the accuracy of the results when a highly elongated finite particle is replaced by an infinite cylinder.

The orientation of the aspherical particles has been measured in interstellar dust clouds and the atmospheres of several planets. A model with infinite cylinders oriented in a certain plane [3,4,6,7] or perfectly oriented along some preferred axis [4,5,7,16] is frequently used in studying the optical properties of such media.

The problem of the transfer of polarized light in anisotropic media is so complex that it is difficult to expect success from analytical solution methods in the general case. Those rare cases when these methods allow the problem to be completely solved are therefore of great value. On the one hand, the exact solutions to the factor equation of transfer can to some extent be used in practice, with some simplifying assumptions; on the other hand, they are a limiting case for more general, strictly posed problems, and are required for testing both numerical and approximate methods.

One such limiting case - an isotropic medium consisting of spherical particles - is the approximation generally used in the interpretation of planetary polarimetric observations. Approximating highly elongated perfectly oriented particles by infinite cylinders allows us to discuss another special case which admits of a simple, exact solution to the vector equation of transfer.

Infinite cylinders of arbitrary cross section have the remarkable property that at large distances, the scattered wave for an electromagnetic wave is incident on the cylinder perpendicular to its axis is cylindrical, the scattering plane is perpendicular to the cylinder axis, and the scattering problem is geometrically two-dimensional, where "two-dimensional" is understood to mean that the characteristics of the scattered light depend on only two coordinates specified in the plane perpendicular to the cylinder axis. If the scattering medium consists of perfectly oriented cylinders, and the light is incident perpendicular to their axes, this two-dimensionality will be preserved in each scattering event, thus making the entire scattering problem for the polarized radiation two-dimensional. Since the scattering plane in this case is always in the same position - perpendicular to the axes of the particles, the phase matrix is identical to the scattering matrix. If the cylinders have circular (or chaotically oriented) cross sections perpendicular to the direction of orientation, the phase matrix, which corresponds to the set of Stokes parameters $\{I_r, I_\perp, U, V\}$, will be of the form [1, Chap. 8; 2, Chap. 15]

$$\hat{F}(\theta) = \begin{bmatrix} \chi_l(\theta) & 0 & 0 & 0 \\ 0 & \chi_r(\theta) & 0 & 0 \\ 0 & 0 & \xi(\theta) & \gamma(\theta) \\ 0 & 0 & -\gamma(\theta) & \xi(\theta) \end{bmatrix}, \quad (2)$$

where θ is the scattering angle. The actual form of the functions in matrix (2) is determined by the index of refraction of the cylinders and the distribution of the cylinders over diameter.

Thus, if we approximate the particles in a medium consisting of highly elongated particles by infinite cylinders, and examine the case when linearly polarized light ($U = V = 0$) is incident perpendicular to the orientation of the particles, the phase matrix will be of the form (32), the scattered radiation will also be linearly polarized, light polarized parallel to the particle axes will be scattered independently from the radiation polarized perpendicular to the axes, and the vector equation of transfer for a plane layer in a three-dimensional medium splits into two independent two-dimensional scalar equations for these components of the radiation; these equations have a rigorous analytical solution in the case of a semi-infinite homogeneous medium. In the general case, when the incident radiation is elliptically polarized, (2) implies that the complete solution of the problem requires another system of two equations of transfer independent of the preceding equations be included in the discussion.

Understandably, the finite length of the real particles means that radiation will not be scattered only in the plane perpendicular to the direction of orientation of the particles. However, the intensity of the scattered radiation will be highest in this plane, and (1) implies that the assumptions made here will be more accurate for the longer, narrower particles.

And so, under the assumptions mentioned above, the transfer of polarized light in a medium consisting of highly elongated particles when linearly polarized light is incident from the exterior in a plane perpendicular to the direction of orientation of the particles is described by two independent two-dimensional scalar equations of transfer:

$$\frac{dI_j}{ds} = -\alpha_j I_j + \epsilon_j, \quad j = l, r,$$

where α_j and ϵ_j are, respectively, the extinction and emission coefficients. The subscripts l and r correspond to light polarized parallel and perpendicular to the orientation of the particles, respectively. These equations take the following form in the case of a plane-parallel medium:

$$\cos \theta \frac{dI_j(\tau_j, \theta)}{d\tau_j} = -I_j(\tau_j, \theta) + B_j(\tau_j, \theta), \quad (3)$$

where τ_j is optical depth and $B_j(\tau_j, \theta)$ is the source function. It is assumed that the boundaries of the medium are parallel to the particle axes; the convention for the angles is shown in Fig. 1, whose plane is perpendicular to the orientation of the particles, and thus lies in the plane of scattering. We write the following expression for the source function

$$B_j(\tau_j, \theta) = \frac{\lambda_j}{2\pi} \int_{-\pi}^{\pi} I_j(\tau_j, \theta') \chi_j(\theta - \theta') d\theta' + B_j^*(\tau_j, \theta), \quad (4)$$

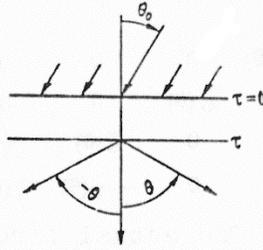


Fig. 1. Coordinates in the plane perpendicular to the direction of orientation of the particles.

where $\lambda_j = \sigma_j/\alpha_j$ is the single scattering albedo, the σ_j are the scattering coefficients, and the $\chi_j(\theta)$ are the scattering indicatrices, i.e., the elements of scattering matrix (2), which have the following normalization condition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \chi_j(\theta) d\theta = 1. \quad (5)$$

The functions $B_j^*(\tau_j, \theta)$ describe the internal radiation sources. Formulae for calculating λ_j , α_j , and the functions $\chi_j(\theta)$ for infinite circular cylinders with arbitrary complex index of refraction are given in [2, Ch. 15; 1, Ch. 8]. A program appropriate for computer calculation is presented in [1, App. B].

The methods developed for solving the usual three-dimensional equation of transfer (see, for example, [12,18]) are entirely suitable for solving equations (3) and (4). It is even simpler to obtain the solution to this particular form of the vector equation than it is for the usual scalar equation, since the scattering problem is planar, and it is not necessary to separate out the azimuthal dependence by expanding the scattering indicatrix χ_j in series of Legendre polynomials and using the superposition theorem for spherical functions.

We shall now present the basic formulae for the case of a semi-infinite homogeneous medium. In doing so, we shall generally omit the derivations of the formulae, since they reduce to an almost complete repetition of the corresponding computations for obtaining the analogous expressions for a three-dimensional medium in the scalar case.

COLLIMATED INCIDENT BEAM

Suppose that a collimated beam of linearly polarized light πS_j ($j = l, r$) is incident on the boundary of a homogeneous, semi-infinite medium consisting of completely oriented cylindrical particles with infinite length from the exterior at an angle $\theta_0 \in [-\pi/2, \pi/2]$. The source function is then given by equation (4), where $B_j^*(\tau_j, \theta, \theta_0) = \frac{\lambda_j}{2} S_j \chi_j(\theta - \theta_0) \exp(-\tau_j / \cos \theta_0)$. After determining the matrix for the reflection of light by the semi-infinite medium using the relations

$$\hat{\rho}(\theta, \theta_0) = \begin{bmatrix} \rho_l(\theta, \theta_0) & 0 \\ 0 & \rho_r(\theta, \theta_0) \end{bmatrix}, \quad (6)$$

$$S_j \rho_j(\theta, \theta_0) \cos \theta_0 = I_j(0, \pi - \theta, \theta_0), \quad \theta \in [-\pi/2, \pi/2],$$

we find that (where the subscript j has been omitted for simplicity)

$$S\rho(\theta, \theta_0)(\cos \theta + \cos \theta_0) = B(0, \pi - \theta, \theta_0) + \frac{\cos \theta}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta') B(0, \theta', \theta_0) d\theta', \quad (7)$$

$$B(0, \theta, \theta_0) = \frac{\lambda}{2} S \left[\chi(\theta - \theta_0) + \frac{\cos \theta_0}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta', \theta_0) \chi(\theta + \theta' + \pi) d\theta' \right]. \quad (8)$$

Substituting (8) into (7), we obtain the following two-dimensional analog of the Ambartsumyan integral equation:

$$\begin{aligned} \rho(\theta, \theta_0)(\cos \theta + \cos \theta_0) = & \frac{\lambda}{2} \left[\chi(\theta + \theta_0 + \pi) + \frac{\cos \theta_0}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta', \theta_0) \chi(\theta' - \theta) d\theta' + \right. \\ & + \frac{\cos \theta}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta') \chi(\theta' - \theta_0) d\theta' + \frac{\cos \theta \cos \theta_0}{\pi^2} \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta') d\theta' \times \\ & \left. \times \int_{-\pi/2}^{\pi/2} \rho(\theta'', \theta_0) \chi(\theta'' + \theta' + \pi) d\theta'' \right]. \end{aligned} \quad (9)$$

The following symmetry relations are valid for the elements of the reflection matrix: $\rho(\theta, \theta_0) = \rho(\theta_0, \theta) = \rho(-\theta, -\theta_0)$, $\theta, \theta_0 \in [-\pi/2, \pi/2]$.

Equation (9) may be solved (for example) by iteration. Using equation (30) for conservative or nearly conservative scattering enables the rate of convergence of the iterative process to be increased (15).

TWO-DIMENSIONAL INFINITE MEDIUM

Suppose that we have an infinite two-dimensional medium with sources of infinite power located at minus infinity. The equation of transfer in this case (as before, we omit the subscript j) takes the following form:

$$\cos \theta \frac{dI_\infty(\tau, \theta)}{d\tau} = -I_\infty(\tau, \theta) + \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} I_\infty(\tau, \theta') \chi(\theta - \theta') d\theta'. \quad (10)$$

the condition $I_\infty(\tau, \theta) \rightarrow 0$ as $\tau \rightarrow \infty$ must be satisfied in nonconservative scattering. The main solution mode for equation (10) is given by the expression

$$I_\infty(\tau, \theta) = i(\theta) \exp(-k\tau), \quad (11)$$

where the function $i(\theta) = i(-\theta)$

$$i(\theta)(1 - k \cos \theta) = \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} i(\theta') \chi(\theta - \theta') d\theta', \quad (12)$$

and the diffusion index k is the minimum root of characteristic equation (12) on the interval $(0, 1]$. Further, it is assumed that

$$\frac{\lambda}{\pi} \int_0^{\pi} i(\theta) d\theta = 1. \quad (13)$$

Equations (5) and (12) imply that $k = 0$ and $i(\theta) \equiv 1$ for conservative scattering ($\lambda = 1$) and an arbitrary scattering indicatrix.

For isotropic scattering, we find from equation (12) that

$$i(\theta) = (1 - k \cos \theta)^{-1}. \quad (14)$$

Substituting (14) into (13) yields

$$k = \sqrt{1 - \lambda^2}. \quad (15)$$

Thus, there is an explicit expression for the diffusion exponent for isotropic scattering in a two-dimensional medium, just as there is in the case of a one-dimensional medium (where $k = \sqrt{1 - \lambda}$).

Writing the scattering indicatrix in the form of a Fourier series expansion, we have

$$\chi(\cos \theta) = 1 + 2 \sum_{n=1}^{\infty} x_n \cos n\theta, \quad (16)$$

where

$$x_n = \frac{1}{\pi} \int_0^{\pi} \chi(\cos \theta) \cos n\theta d\theta, \quad (17)$$

and we always have $|x_n| < 1$. From (12) and (16) we find that

$$i(\theta)(1 - k \cos \theta) = 1 + 2 \sum_{n=1}^{\infty} x_n c_n \cos n\theta, \quad (18)$$

where

$$c_n = \frac{\lambda}{\pi} \int_0^{\pi} i(\theta) \cos n\theta d\theta. \quad (19)$$

Using the addition formulae for the trigonometric formulae, we may easily obtain a recurrence relation for the c_n from (18):

$$c_{n+1} + c_{n-1} = 2(1 - \lambda x_n) c_n / k, \quad (20)$$

with $c_0 = 1$ and $c_1 = (1 - \lambda)/k$. Using (20), the characteristic equation for determining k may be written in the form of the following continued fraction:

$$1 - \lambda = \frac{k^2}{2(1 - \lambda x_1) - \frac{k^2}{2(1 - \lambda x_2) - \dots}}. \quad (21)$$

It may be shown that if the scattering matrix can be represented by only n terms in the Fourier series expansion, the characteristic equation is then an n -th degree polynomial in k^2 .

In nearly conservative scattering ($1 - \lambda \ll 1$), (18), (20), and (21) imply the asymptotic representations

$$k = \sqrt{2(1-\lambda)(1-x_1)} \left\{ 1 + \frac{1}{2} \left(\frac{x_1}{1-x_1} - \frac{1}{2(1-x_2)} \right) (1-\lambda) + O[(1-\lambda)^2] \right\}, \quad (22)$$

$$i(\theta) = 1 + \sqrt{\frac{1-\lambda}{1-x_1}} \cos \theta + \left(1 + \frac{\cos 2\theta}{1-x_2} \right) (1-\lambda) + O[(1-\lambda)^{3/2}]. \quad (23)$$

INVARIANCE RELATIONS

On the basis of the generalized invariance principle [10,11], we may write for all

$$I(\tau + \tau_1, \theta, \theta_0) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I(\tau_1, \theta', \theta_0) I(\tau, \theta, \theta') d\theta' + I(\tau, \theta, \theta_0) \times \\ \times \exp(-\tau_1/\cos \theta_0) + I(\tau_1, \theta, \theta_0) \exp(-\tau/\cos \theta) \Theta\left(\frac{\pi}{2} - |\theta|\right), \quad (24)$$

where $\Theta(x)$ is the unit step function. Equation (24) immediately implies the doubling formula [11]

$$I(2\tau, \theta, \theta_0) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I(\tau, \theta', \theta_0) I(\tau, \theta, \theta') d\theta' + \\ + I(\tau, \theta, \theta_0) \left[\exp(-\tau/\cos \theta_0) + \exp(-\tau/\cos \theta) \Theta\left(\frac{\pi}{2} - |\theta|\right) \right], \quad (25)$$

which leads to a simple algorithm for numerical calculation of the internal radiation field [8]. The following asymptotic expression for the radiation intensity in the deep layers of the medium ($\tau \gg 1$) [12, Ch. II] must be kept in mind:

$$I^{as}(\tau, \theta, \theta_0) = i(\theta) u(\theta_0) \cos \theta_0 \exp(-k\tau), \quad (26)$$

where the function $u(\theta_0)$ (the boundary value for the intensity in the Milne problem) is normalized according to the condition

$$\frac{2}{\pi} \int_0^{\pi/2} u(\theta) i(\theta) \cos \theta d\theta = 1. \quad (27)$$

Treating the upper and lower half-spaces in the infinite medium separately, we obtain the following two invariance relations ($\theta \in [-\pi, \pi]$) [11] when (11) is taken into account:

$$i(\theta) \exp(-k\tau) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I(\tau, \theta, \theta') i(\theta') d\theta' + i(\theta) \exp(-\tau/\cos \theta) \Theta\left(\frac{\pi}{2} - |\theta|\right), \quad (28)$$

$$i(\pi - \theta) \exp(k\tau) = MI(\tau, \theta) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I(\tau, \theta, \theta') i(\pi - \theta') d\theta' + \\ + i(\pi - \theta) \exp(-\tau/\cos \theta) \Theta\left(\frac{\pi}{2} - |\theta|\right), \quad (29)$$

where $I(\tau, \theta)$ is the radiation intensity in the Milne problem and M is a normalization factor. Setting $\tau = 0$ in (28) and (29), and taking into account the fact that by definition, $u(\theta) = I(0, \pi - \theta)$, $\theta \in [-\pi/2, \pi/2]$, we obtain two important analogs of the Van de Hulst-Sobolev relations [18, Ch. 5; 12, Ch. II]:

$$i(\pi - \theta) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta') i(\theta') \cos \theta' d\theta', \quad (30)$$

$$i(\theta) = M\mu(\theta) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \rho(\theta, \theta') i(\pi - \theta') \cos \theta' d\theta'. \quad (31)$$

Taking (27) into account, we find from (30) and (31) that

$$M = \frac{2}{\pi} \int_0^{\pi} i^2(\theta) \cos \theta d\theta. \quad (32)$$

For nearly conservative scattering, this quantity is (as implied by (32) and (33) given by the expression

$$M = 2 \sqrt{\frac{1-\lambda}{1-x_1}} + O[(1-\lambda)^{3/2}]. \quad (33)$$

Using equations (28) and (29) and taking (22), (23), and (33) into account allows us to obtain the conservative limit of expression (29) [11]:

$$I_0(\tau, \theta) = \sqrt{2}(1-x_1)\tau + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I_0(\tau, \theta, \theta') \cos \theta' d\theta' - \left[1 - \exp(-\tau/\cos \theta) \Theta\left(\frac{\pi}{2} - |\theta|\right) \right] \cos \theta. \quad (34)$$

Both here and below, we shall denote values under conservative scattering by the subscript "0."

An analog of Chandrasekhar's formula [13, §29.3] follows from (34) with $\tau = 0$:

$$\mu_0(\theta) = \cos \theta + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \rho_0(\theta, \theta') \cos^2 \theta' d\theta'. \quad (35)$$

Finally, using an analog of Chandrasekhar's invariance relation [13, §29.1]:

$$I(\tau, \pi - \theta, \theta_0) = \rho(\theta, \theta_0) \exp(-\tau/\cos \theta_0) \cos \theta_0 + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} I(\tau, \theta', \theta_0) \rho(\theta, \theta') \cos \theta' d\theta', \quad (36)$$

which follows from (24) with $\tau = 0$, and expressions (30) and (31), we obtain the following intensity-squared integrals of the equation of transfer in a two-dimensional medium [14] ($\tau, \tau_1 \geq 0$ and $\theta, \theta_1 \in [-\pi/2, \pi/2]$):

$$\frac{1}{\pi} \int_{-\pi}^{\pi} I(\tau, \pi - \theta', \theta) I(\tau_1, \theta', \theta_1) \cos \theta' d\theta' = I(\tau_1, \pi - \theta, \theta_1) \exp(-\tau/\cos \theta) \cos \theta - I(\tau, \pi - \theta_1, \theta) \exp(-\tau_1/\cos \theta_1) \cos \theta_1, \quad (37)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} I(\tau, \theta', \theta) i(\pi - \theta') \cos \theta' d\theta' = -i(\pi - \theta) \exp(-\tau/\cos \theta) \cos \theta, \quad (38)$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} I(\tau, \theta', \theta) i(\theta') \cos \theta' d\theta' = [M\mu(\theta) \exp(-k\tau) - i(\theta) \exp(-\tau/\cos \theta)] \cos \theta. \quad (39)$$

We shall also present two formulas for the case of nearly conservative scattering ($1 - \lambda \ll 1$). By analogy with the three-dimensional case, we have [12, Ch. II]:

$$\rho(\theta, \theta_0) = \rho_0(\theta, \theta_0) - \sqrt{\frac{1-\lambda}{1-x_1}} u_0(\theta) u_0(\theta_0) + O(1-\lambda), \quad (40)$$

$$u(\theta) = u_0(\theta) \left(1 - \gamma_0 \sqrt{\frac{1-\lambda}{1-x_1}} \right) + O(1-\lambda), \quad (41)$$

where

$$\gamma_0 = \frac{2}{\pi} \int_0^{\pi/2} u_0(\theta) \cos^2 \theta d\theta. \quad (42)$$

Thus, we have obtained all of the necessary relations which will allow us to determine the radiation field in a medium consisting of completely oriented highly elongated particles illuminated from the outside by linearly polarized light perpendicular to the orientation of the particles. This problem may also be solved by another, now classical, method due to Sobolev [12, Ch. V, VI].

In conclusion, we should note that although the two-dimensional scalar equation of transfer was obtained in discussing a particular physical problem, its range of application may turn out to be even broader. First of all, the two-dimensional approximation allows one to take the angular structure of the radiation into account and is thus not subject to the main drawback of the one-dimensional approximation to the equation of transfer, which is, because of its simplicity, widely used in many branches of physics. Second, there are also other problems in physics with two dimensional geometry that lead to similar equations. In particular, as it turns out, the two-dimensional integral for the source function in the case of isotropic scattering is identical to the edge-diffraction equation for electromagnetic waves and thus makes physical sense in a certain way. This is why, in our view, the scalar two-dimensional equation of transfer is worthy of study on its own merits.

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