Arbitrary Lagrangian-Eulerian Schemes for Ocean Modelling

&

A Few Memories of Unstructured Mesh Methods for CFD

Darren Engwirda

Massachusetts Institute of Technology
NASA Goddard Institute for Space Studies
‘Conventional’ CFD differs from GFD in a number of important ways:

**Pressure Coupling**

Velocity-pressure coupling is ‘isotropic’ – no hydrostatic assumption:
- Non-hydrostatic pressure distribution computed at each time-step.
‘Conventional’ CFD differs from GFD in a number of important ways:

**Pressure Coupling**
Velocity-pressure coupling is ‘isotropic’ – no hydrostatic assumption:
- Non-hydrostatic pressure distribution computed at each time-step.

**Sub-grid Modelling**
Resolve boundary-layer flows through mesh adaptation:
- Sub-grid parameterisations used less frequently.
- Direct Numerical Simulation (DNS) not impossible.
‘Conventional’ CFD differs from GFD in a number of important ways:

### Pressure Coupling

Velocity-pressure coupling is ‘isotropic’ – no hydrostatic assumption:
- Non-hydrostatic pressure distribution computed at each time-step.

### Sub-grid Modelling

Resolve boundary-layer flows through mesh adaptation:
- Sub-grid parameterisations used less frequently.
- Direct Numerical Simulation (DNS) not impossible.

### Geometric Constraints

Solve flow problems for arbitrarily complex geometries.
- Use unstructured meshes and numerical methods.
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Ubiquitous Aerodynamics Studies
1. CFD – Voronoi-based Finite Volume Schemes

Integrate equations of motion in divergence form over control volumes:

\[ \int_{\Omega} \frac{dq}{dt} + \nabla \cdot (F(q)) - S_q \, dV = 0 \]

- A Voronoi diagram is a set of polygonal cells.
- Each cell contains varying numbers of edges.
- The edges of each cell are always \textbf{orthogonal} to a common centre.
- The Voronoi diagram is constructed upon an underlying triangulation.
1. CFD – Voronoi Finite-Volumes

Variable resolution Voronoi mesh, clustering elements in boundary layer regions.
2. Mesh – Unstructured Triangulations

The creation of ‘optimal’ unstructured triangulations & Voronoi diagrams is non-trivial:

- Need to ensure that ‘element-quality’ is adequate:
  - Don’t want highly skewed cells – aim for equilateral triangles.
  - Don’t want cell size to vary too rapidly.

- Need to optimise both vertex positions and mesh topology.
2. Mesh – Unstructured Triangulations

The creation of ‘optimal’ unstructured triangulations & Voronoi diagrams is non-trivial:

- Need to ensure that ‘element-quality’ is adequate:
  - Don’t want highly skewed cells – aim for equilateral triangles.
  - Don’t want cell size to vary too rapidly.

- Need to optimise both vertex positions and mesh topology.

The so-called Delaunay Triangulation offers a convenient framework for mesh generation. Given a set of vertices $X \subset \mathbb{R}^d$, the Delaunay triangulation $\mathcal{T} = \text{Del}(X)$ is known to be ‘optimal’ for a range of geometric criteria.
2. Mesh – Quality Delaunay Triangulations

‘Refinement’ algorithms incrementally add vertices to a coarse mesh until all constraints are satisfied:

- A coarse triangulation is built based on the external geometry of the domain.

In $\mathbb{R}^2$, the refinement algorithm can achieve a minimum angle $\theta_{\text{min}} \geq 30^\circ$. 

![Diagram of a mesh with quality Delaunay triangulations]
2. Mesh – Quality Delaunay Triangulations

‘Refinement’ algorithms incrementally add vertices to a coarse mesh until all constraints are satisfied:

- A coarse triangulation is built based on the external geometry of the domain.
- Additional vertices are added to ‘remove’ any poor quality triangles by splitting them.
‘Refinement’ algorithms incrementally add vertices to a coarse mesh until all constraints are satisfied:

- A coarse triangulation is built based on the external geometry of the domain.
- Additional vertices are added to ‘remove’ any poor quality triangles by splitting them.
- All elements in the final mesh satisfy shape and size constraints. In $\mathbb{R}^2$, the refinement algorithm can achieve a minimum angle $\theta_{\text{min}} \geq 30^\circ$. 

2. Mesh – Quality Delaunay Triangulations
Surface and volumetric triangulations of a turbine blade for a 3d CFD study.
Unstructured GFD Applications?
Unstructured GFD Applications?
II.
Equations of motion can be represented in either an Eulerian or Lagrangian form:

- **Eulerian Form**: Mesh is fixed and transport is achieved through explicit evaluation of cell-wise fluxes.

- **Lagrangian Form**: Mesh moves with the flow. Flux evaluation is replaced by mesh movement.
Equations of motion can be represented in either an Eulerian or Lagrangian form:

- **Eulerian Form**: Mesh is fixed and transport is achieved through explicit evaluation of cell-wise fluxes.

- **Lagrangian Form**: Mesh moves with the flow. Flux evaluation is replaced by mesh movement.

Lagrangian methods allow the mesh to align locally with features of the flow:

**Quasi-Isopycnal Representation**

Lagrangian vertical transport can be used to achieve a quasi-isopycnal representation in the open-ocean, where the flow is essentially adiabatic.

**Minimisation of spurious vertical mixing.**
3. ALE – Layered Vertical Structure

The aim is to follow the approach of HYCOM, introducing a ‘flexible’ vertical discretisation that:

- Follows isopycnals where possible.
- Smoothly transitions to other representations where necessary. (z-model in mixed layer, $\sigma$-model near sharp topography, etc).

\[1\] Temperature profiles from Bleck 2004
The equations of motion for the ocean can be written as a set of layer-wise conservation laws:

\[
\frac{d\mathbf{u}_h}{dt} + \nabla \cdot (\mathbf{u}_h \mathbf{u}_h^T) = -\nabla_p (\Phi) + \mathbf{S}_{u_h}
\]

\[
\frac{d\Phi}{dp} = -\alpha
\]

\[
\frac{d\Delta p}{dt} + \nabla \cdot (\mathbf{u}_h \Delta p) = \mathbf{S}_p
\]

\[
\frac{d\theta, S}{dt} + \nabla \cdot (\mathbf{u}_h \theta, S) = \mathbf{S}_{\theta, S}
\]

Rather than introducing a ‘hybrid’ vertical coordinate (as per HYCOM), we instead form a finite-volume scheme, integrating over layers of variable thickness.
Issues can arise with purely-Lagrangian methods due to the movement of the grid:

- The grid may become overly distorted due to local flow characteristics.
- The grid may evolve into a non-optimal configuration.
Issues can arise with purely-Lagrangian methods due to the movement of the grid:

- The grid may become overly distorted due to local flow characteristics.
- The grid may evolve into a non-optimal configuration.

These issues can be mitigated through use of an Arbitrary Lagrangian Eulerian (ALE) approach:

**Quasi-Eulerian Re-mapping**

If the grid is ‘far-enough’ away from optimal, **re-map** all flow variables onto a new target grid via interpolation.
3. ALE – Simple Sketch of an ALE Algorithm

\[
\Phi^t = \Phi_b - \int_{\rho_b}^{p_t} \alpha(\theta^t, S^t, p^t) \, dp
\]

\[
\Delta p \mathbf{u}_h^{t+\delta t} = \mathbf{u}_h^t + \delta t(-\nabla_p \Phi^t - \nabla \cdot (\Delta p \mathbf{u}_h \mathbf{u}_h^T)^t + \Delta p \mathbf{S}_u)
\]

\[
\Delta p^{t+\delta t} = \Delta p^t - \delta t \nabla \cdot (\Delta p \mathbf{u}_h^{t+\delta t})
\]

\[
(\Delta p \theta, \Delta p S)^{t+\delta t} = (\Delta p \theta, \Delta p S)^t - \delta t \nabla \cdot (\Delta p \mathbf{u}_h^{t+\delta t} (\theta^t, S^t))
\]
3. ALE – Simple Sketch of an ALE Algorithm

\[ \Phi^t = \Phi_b - \int_{\rho_b}^{\rho_t} \alpha(\theta^t, S^t, p^t) \, dp \]

\[ \Delta p u_{h}^{t+\delta t} = u_{h}^{t} + \delta t(-\nabla_p \Phi^t - \nabla \cdot (\Delta p u_{h} u_{h}^T)^t + \Delta p S_u) \]

\[ \Delta p^{t+\delta t} = \Delta p^t - \delta t \nabla \cdot (\Delta p u_{h}^{t+\delta t}) \]

\[ (\Delta p \theta, \Delta p S)^{t+\delta t} = (\Delta p \theta, \Delta p S)^t - \delta t \nabla \cdot (\Delta p u_{h}^{t+\delta t} (\theta^t, S^t)) \]

At \( t + \delta t \) the grid has drifted (due to vertical transport):

If the grid is not where we want it, we can re-map all flow variables onto a new grid via a (conservative) interpolation scheme.
3. ALE – Column-wise Sketch of an ALE Algorithm

- **Update** flow variables and cell thickness via Lagrangian motion.
- **Reconstruct** cell-wise polynomials on current mesh.
- **Integrate** polynomials over new mesh to get new cell means.

\[ a \]

\[ a \text{from Adcroft 2013} \]
Considering that the $\Delta p$ layers are sloping and non-uniform in thickness, evaluation of the pressure gradient term $\nabla_p(\Phi)$ is non-trivial.
3. ALE – Evaluating the Pressure Gradient Force

Considering that the $\Delta p$ layers are sloping and non-uniform in thickness, evaluation of the pressure gradient term $\nabla_p(\Phi)$ is non-trivial.

A well-known approach approximates the pressure gradient on a sloping layer ‘s’ directly, as a finite-difference of the Montgomery potential $M$:

$$M = \alpha \nabla_s(p) + \nabla_s \Phi$$
Considering that the $\Delta p$ layers are sloping and non-uniform in thickness, evaluation of the pressure gradient term $\nabla_p(\Phi)$ is non-trivial.

A well-known approach approximates the pressure gradient on a sloping layer ‘s’ directly, as a finite-difference of the Montgomery potential $M$:

$$M = \alpha \nabla_s(p) + \nabla_s \Phi$$

Due to non-linearities in the equation of state $\alpha(\theta, S, p)$, such an approach is not typically stable. In regions of sharp topography and stratification:

- A small fraction of the vertical force balance can ‘contaminate’ the horizontal.
- Such occurrences can cause spurious ‘spontaneous motion’ from an equilibrated state.
Following an approach of Adcroft et al. [1], the pressure gradient can instead be evaluated *indirectly*, via a finite-volume integral:
Following an approach of Adcroft et al. [1], the pressure gradient can instead be evaluated \textit{indirectly}, via a finite-volume integral:

\[ \int_{\Omega} \nabla p(\Phi) \, dp \, dx = \oint_{\partial \Omega} \Phi \, dC \]
3. ALE – Evaluating the Pressure Gradient Force

Following an approach of Adcroft et al. [1], the pressure gradient can instead be evaluated **indirectly**, via a finite-volume integral:

\[
\int_{\Omega} \nabla_p(\Phi) \, dp \, dx = \oint_{\partial \Omega} \Phi \, dC
\]

This formulation accounts for the fully non-linear distribution of \( \Phi \) around each element:
3. ALE – Evaluating the Pressure Gradient Force

Assess spurious motion with variable topography, linear stratification.
3. ALE – Evaluating the Pressure Gradient Force

Assess spurious motion with variable topography, linear stratification.
Assess spurious motion with variable topography, linear stratification.
Given a sufficiently high-order quadrature, the finite-volume pressure gradient formulation achieves $\approx 0.0$ error.

Such a scheme allows flexible vertical discretisation, but will maintain equilibrium in the presence of sharp topography and stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, Δ10° temperature stratification.
Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, Δ10° temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
3. ALE – Initial Results: Dense Overflow

Channel: 200km wide, 2000m deep, $\Delta 10^\circ$ temperature stratification.
4. Summary

- Developed a simple ‘proof-of-concept’ layered ocean model using ALE methodologies.

- Developed a stable pressure gradient formulation that minimises pressure gradient errors with arbitrary layer geometries/stratification.

- Looking to improve 2D model:
  - Variable number of layers per column.
  - General boundary conditions.
  - Sub-grid parameterisations.

- Incorporate ALE technology into the next iteration of the GISS ocean model.