Solving 1-D advection-diffusion equations

Timothy M. Hall
NASA GISS

Abstract

Here are some notes on solving 1-D advection-diffusion equations in various guises. These types of equations often arise in idealized analyses of tracers and first-order interpretations of observations in various fields of the earth sciences. All of this is "standard" and can be found in various textbooks, but it may be useful to have the techniques collected in one place. I plan to add to this from time to time.

1. General Equation

The general advection-diffusion equation (ADE) with constant coefficients for tracer mixing ratio $q = q(x,t)$ is

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} - k\frac{\partial^2 q}{\partial x^2} = S \tag{1}$$

where u is the velocity, k the diffusivity, and S a source term. Boundary conditions are applied on q ("Dirichlet" condition), or dq/dx ("Neumann" condition), or linear combination of both ("mixed" condition). The equation is second order, so two independent boundary conditions must be specified.

In terms of non-dimensional x and t, the equation is

$$\frac{\partial q}{\partial t} + \frac{\partial q}{\partial x} - \frac{\partial^2 q}{\partial x^2} = S \tag{2}$$

Afterwords, dimensions can then be returned by $x \to ux/k$ and $t \to kt/u^2$.

Note that the ADE is equivalent to the diffusion equation with exponentially decreasing background density as used by Hall and Plumb. Simply substitute $u = k/H$ where H is the density scale height.

2. Age Spectrum

The age spectrum is the solution with $s = 0$ and time-dependent boundary condition $q(0,t) = \delta(t)$. (The second condition is that q cannot grow exponentially in space.) The solution strategy is to use the Laplace tranform to convert the PDE (time and space) to an ODE (space), solve the ODE in the transform domain, and then perform (or look up) the inverse Laplace transform to get back to the time domain. The Laplace transform of q is defined as

$$\tilde{q}(x,s) = \int_0^\infty e^{-st} q(x,t)dt \tag{3}$$

where s is the transform variable. Note that the Laplace transform of $\partial q/\partial t$ is $s\tilde{q}$. Therefore, the transform of the dimensionless ADE with $S = 0$ is

$$s\tilde{q} + \frac{d\tilde{q}}{dx} - \frac{d^2\tilde{q}}{dx^2} = 0. \tag{4}$$

which is an ODE. The appropriate boundary condition is the Laplace transform of the time-domain condition $q(0,t) = \delta(t)$, which is $\tilde{q}(0,s) = 1$. In addition there remains the condition of no exponential growth in x.

Solutions to this ODE are of the form $\tilde{q} = e^{\lambda x}$. Substituting yields the "characteristic equation" $s + \lambda - \lambda^2 = 0$, which has the solution

$$\lambda_\pm = \frac{1}{2} \pm \sqrt{s + \frac{1}{4}} \tag{5}$$

Only the negative solution is allowed to keep growth bounded at $x = \infty$. This leaves the solution $\tilde{q} = Ae^{\lambda_- x}$, for general constant A. Applying the BC $\tilde{q}(0,s) = 1$ implies $A = 1$. All told, then, the Laplace tranform of the dimensionless age spectral solution is

$$\tilde{q}(x,s) = e^{x/2} e^{-x\sqrt{s+1/4}} \tag{6}$$

The inverse Laplace transform is obtained by the contour integration (a "Bromwich" integral)

$$q(x,t) = L^{-1}(\tilde{q}(x,s)) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}\tilde{q}(x,s)ds. \tag{7}$$

Reasonable tables of Laplace transforms and their inverses (e.g., Abromowitz and Stegun) include the cases likely to be encountered for 1-D ADE for fairly simple boundary conditions. Using the "shift" property $L^{-1}(\tilde{q}(s+a)) = e^{-at}L^{-1}(\tilde{q}(s))$, leaves the inverse transform of $e^{-x\sqrt{s}}$. Using the inverse transform from Abromowitz and Stegun, and rearranging terms in the exponent, yields

$$q(x,t) = \frac{x}{\sqrt{4\pi t^3}} e^{-(x-t)^2/4t} \tag{8}$$

Replacing dimensions, and using $\mathcal{G}$ for the age spectrum, this is

$$\mathcal{G}(x,t) = \frac{x}{\sqrt{4\pi k t^3}} e^{-(x-ut)^2/4kt} \tag{9}$$

3. Simple "radioactive" loss term

The solution to the ADE can be extended at no cost to include a loss term of the form $-\lambda q$; i.e.,

$$\frac{\partial q}{\partial t} + \ \ldots \ = -\lambda q \tag{10}$$

when λ is constant and uniform (e.g., radioactive decay of a chemically inert tracer, or dilution of a 1-D column by relaxation to a suroundiing infinite reservoir with no tracer). Define $q' = qe^{\lambda t}$, substitute, and find an equation for q' the same as the conserved tracer.

4. Explicit source, unbounded domain

The textbook Green's function is the solution to an explicit point source (as opposed to an impulse boundary condition on mixing ratio, as treated above). That is, in equation (2) the source term $S(x,t) = \delta(x-x_0)\delta(t-t_0)$. Taking $x_0 = 0$ and $t_0 = 0$, and taking the Laplace transform gives

$$s\tilde{q} + \frac{d\tilde{q}}{dx} - \frac{d^2\tilde{q}}{dx^2} = \delta(x) \tag{11}$$

First, we treat the "unbounded" case; that is, the boundary conditions are that there is no exponential growth at $\pm\infty$. For $x \neq 0$, the solutions are again of the from $\tilde{q} = e^{\lambda_{\pm}x}$, with the same characteristic equation for λ as above. For $x > 0$ choose λ_-, and for $x < 0$ choose λ_+, in order to have bounded solutions. This leaves the general solution

$$\tilde{q} = \begin{cases} Ae^{x/2}e^{-x\sqrt{s+1/4}} & x > 0 \\ Be^{-x/2}e^{+x\sqrt{s+1/4}} & x < 0 \end{cases} \tag{12}$$

Two conditions provide constraints for A and B. The first condition is continuity of $\tilde{q}$ at $x = 0$, implying $B = A$. The second condition is revealed by integrating equation (11) from $-\epsilon$ to $+\epsilon$ for small ϵ. By continuity, the first two terms on the left vanish, while the delta function on the left gives unity, leaving

$$\frac{d\tilde{q}}{dx}|_{-\epsilon} - \frac{d\tilde{q}}{dx}|_{+\epsilon} = 1 \tag{13}$$

Substituting the general solution, using $B = A$, and taking the limit $\epsilon \to 0$ gives $A = 1/\sqrt{s+1/4}$. Therefore

$$\tilde{q} = \frac{1}{\sqrt{s+1/4}} \begin{cases} e^{x/2}e^{-x\sqrt{s+1/4}} & x > 0 \\ e^{-x/2}e^{+x\sqrt{s+1/4}} & x < 0 \end{cases} \tag{14}$$

After the "shift" property is used, it remains to obtain the inverse transform of $e^{-|x|\sqrt{s}}/\sqrt{s}$. (Now, there is a pole at the origin.) Using a table, substituting, rearranging terms, and replacing dimensions, yields

$$q(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-ut)^2/4kt} \tag{15}$$

(The separate expressions for $\pm x$ collapse to one in the algebra, because $(|x|+t)^2 = (x-t)^2$ for $x < 0$.)