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REFLECTION OF POLARIZED LIGHT BY PLANE-PARALLEL SLABS CONTAINING RANDOMLY-ORIENTED, NONSPHERICAL PARTICLES

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Abstract—The computation of the Stokes parameters of light reflected by plane-parallel, isotropic slabs composed of randomly-oriented, nonspherical particles of revolution is considered. Accurate numerical results are reported for homogeneous slabs containing monodisperse, randomly-oriented, homogeneous spheroids and Chebyshev particles.

1. INTRODUCTION

Before numerically solving the equation of transfer of polarized light in a scattering medium, the single-scattering properties of an infinitesimal volume element must be determined. For plane-parallel, macroscopically-isotropic media, a sufficient set of quantities specifying the volume element is formed by the albedo for single scattering and the coefficients that appear in expansions of the elements of the scattering matrix in (combinations of) generalized spherical functions.[1–6] Computations of these expansion coefficients for homogeneous and radially inhomogeneous spherical particles are described in Refs. 2 and 7–12. Recently, de Haan[13] used Asano and Yamamoto's[14] solution for homogeneous spheroidal scatterers, as modified by Schaefer,[15] and a method of numerical angle integrations to compute the expansion coefficients for randomly-oriented, homogeneous spheroids. This method of numerical angle integrations is flexible and, in principle, may be used for any nonspherical particles provided that a proper procedure to compute light-scattering properties of a single particle in arbitrary orientation is available. Unfortunately, however, this method implies computation of the elements of the scattering matrix for a representative set of particle orientations and scattering angles which makes it a rather time-consuming process.

In our recent paper,[16] an efficient rigorous method to compute the expansion coefficients for randomly-oriented, axially-symmetric particles has been proposed. As the mathematical and numerical basis, we used Waterman's T-matrix approach,[17–21] which seems to be the most powerful tool for solving light-scattering problems for homogeneous and composite, axially-symmetric particles with sizes not too large compared to a wavelength.[22–28] We have shown that, instead of computing numerically the average scattering properties of a particle ensemble by averaging results for scattering by a single particle with varying orientation, the expansion coefficients can be analytically expressed in some basic quantities that depend only upon the size, morphology, and composition of the scattering particle and do not depend upon any angular variable. These basic quantities are the elements of the T-matrix of the axially-symmetric scatterer calculated with respect to the coordinate system with the z-axis along the axis of symmetry of the scatterer. In a certain sense, our method is an extension of Domke's[7] procedure since, for spherical particles (homogeneous or radially inhomogeneous), our formulae are reduced to those derived in Ref. 7 and rederived in Refs. 9 and 12.

The purpose of the present paper is to use our method for computing the expansion coefficients to generate accurate numerical results for polarized radiation *multiply*-scattered by nonspherical particles of various shapes. In Sec. 2, we recapitulate the main formulae and equations and briefly discuss the computational scheme that was used in our computer calculations. In Sec. 3, the Stokes

parameters for light reflected by homogeneous, plane-parallel slabs are computed for two models of nonspherical scattering particles.

It should be noted here that multiple scattering of polarized light by nonspherical particles was studied in recent publications by Ishimaru et al.[29,30] Nevertheless, unlike our paper, these references deal with the particles that are perfectly aligned rather than randomly oriented.

2. BASIC FORMULAE AND COMPUTATIONAL METHODS

To describe the state of polarization of a beam of light, we use the Stokes parameters I, Q, U, and V, as defined by Hovenier and van der Mee.[5] However, in our paper, we adopt a time dependence for the electrical field of the form $\exp(-i\omega t)$; therefore, our Stokes parameter V will have the opposite sign. The Stokes vector $\mathbf{I}$ is defined as a (4×1) column having the Stokes parameters as its components as follows:

$$\mathbf{I} = (I, Q, U, V)^{\mathrm{T}} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}, \tag{1}$$

where T denotes matrix transposition. To specify the direction of light propagation in a plane-parallel slab, we use the couple (u, φ), where $u\ (-1 \leqslant u \leqslant 1)$ is the cosine of the angle between this direction and the inward normal to the upper boundary of the slab and φ is the azimuth angle, which is measured clockwise when looking upwards. Also, we define $\mu = |u|$.

The upper boundary of the slab is illuminated by a parallel beam of light, which is specified by the Stokes vector $\pi\mathbf{F}_{\mathrm{in}}\ \delta(\mu - \mu_0)\delta(\varphi - \varphi_0)$, where $\delta(x)$ is the Dirac delta function. We first expand the Stokes vector of the reflected light $\mathbf{I}(t; -\mu, \mu_0, \varphi - \varphi_0)$ in a Fourier series, i.e.

$$\mathbf{I}(t; -\mu, \mu_0, \varphi - \varphi_0) = \sum_{m=0}^{\infty} (2 - \delta_{m0})[\mathbf{I}_c^m(t; \mu, \mu_0)\cos m(\varphi - \varphi_0) + \mathbf{I}_s^m(t; \mu, \mu_0)\sin m(\varphi - \varphi_0)], \tag{2}$$

where t is the optical thickness of the slab and δ_{ij} is the Kronecker delta. We then have[4,5,31,32]

$$\mathbf{I}_c^m(t; \mu, \mu_0) = \tfrac{1}{2}(\mathbf{E} + \mathbf{D})\mathbf{I}_m^+(t; \mu, \mu_0) + \tfrac{1}{2}(\mathbf{E} - \mathbf{D})\mathbf{I}_m^-(t; \mu, \mu_0), \tag{3}$$

$$\mathbf{I}_s^m(t; \mu, \mu_0) = \tfrac{1}{2}(\mathbf{E} - \mathbf{D})\mathbf{I}_m^+(t; \mu, \mu_0) - \tfrac{1}{2}(\mathbf{E} + \mathbf{D})\mathbf{I}_m^-(t; \mu, \mu_0), \tag{4}$$

where

$$\mathbf{I}_m^{\pm}(t; \mu, \mu_0) = \mu_0\mathbf{R}^m(t; \mu, \mu_0)\mathbf{F}^{\pm}, \tag{5}$$

$$\mathbf{F}^{\pm} = \tfrac{1}{2}(\mathbf{E} \pm \mathbf{D})\mathbf{F}_{\mathrm{in}}, \tag{6}$$

$$\mathbf{E} = \mathrm{diag}(1, 1, 1, 1), \tag{7}$$

and

$$\mathbf{D} = \mathrm{diag}(1, 1, -1, -1). \tag{8}$$

The (4×4) matrices $\mathbf{R}^m(t; \mu, \mu_0)$ are solutions of the invariant imbedding equation[33,34]

$$\begin{aligned} \frac{\mathrm{d}\mathbf{R}^m(t; \mu, \mu_0)}{\mathrm{d}t} = &-\left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\mathbf{R}^m(t; \mu, \mu_0) + \frac{w}{4\mu\mu_0}\mathbf{Z}^m(-\mu, \mu_0) \\ &+ \frac{w}{2\mu_0}\int_0^1 \mathrm{d}\mu'\,\mathbf{R}^m(t; \mu, \mu')\mathbf{Z}^m(\mu', \mu_0) + \frac{w}{2\mu}\int_0^1 \mathrm{d}\mu'\,\mathbf{Z}^m(-\mu, -\mu')\mathbf{R}^m(t; \mu', \mu_0) \\ &+ w\int_0^1\int_0^1 \mathrm{d}\mu'\,\mathrm{d}\mu''\,\mathbf{R}^m(t; \mu, \mu')\mathbf{Z}^m(\mu', -\mu'')\mathbf{R}^m(t; \mu'', \mu_0), \end{aligned} \tag{9}$$

which describes how the Fourier components of the reflection matrix $\mathbf{R}^m(t; \mu, \mu_0)$ are changed when a new, optically-thin layer, which is specified by the single scattering albedo w and the matrices $\mathbf{Z}^m(u, u')$, is added to the top of the slab. Equation (9) is supplemented by the initial condition

$$\mathbf{R}^m(0; \mu, \mu_0) = \mathbf{R}^m(\mu, \mu_0). \tag{10}$$

The (4×4) matrices $\mathbf{Z}^m(u, u')$ are defined by[4,5,31,32]

$$\mathbf{Z}^m(u, u') = (-1)^m \sum_{s=m}^{\infty} \mathbf{P}^s_m(u)\mathbf{S}^s\mathbf{P}^s_m(u'), \tag{11}$$

where the matrices $\mathbf{P}^s_m(u)$ are given by the formulae

$$\mathbf{P}^s_m(u) = \begin{bmatrix} P^s_{m0}(u) & 0 & 0 & 0 \\ 0 & P^s_{m+}(u) & P^s_{m-}(u) & 0 \\ 0 & P^s_{m-}(u) & P^s_{m+}(u) & 0 \\ 0 & 0 & 0 & P^s_{mo}(u) \end{bmatrix} \tag{12}$$

and

$$P^s_{m\pm}(u) = \tfrac{1}{2}[P^s_{m-2}(u) \pm P^s_{m2}(u)]. \tag{13}$$

In Eqs. (12) and (13), $P^s_{mn}(u)$ are generalized spherical functions, which have been defined and extensively studied by Gel'fand et al[35] (the principal properties of these functions are summarized in Ref. 5). The matrices $\mathbf{S}^s$ have the form

$$\mathbf{S}^s = \begin{bmatrix} a^s_1 & b^s_1 & 0 & 0 \\ b^s_1 & a^s_2 & 0 & 0 \\ 0 & 0 & a^s_3 & b^s_2 \\ 0 & 0 & -b^s_2 & a^s_4 \end{bmatrix}. \tag{14}$$

The elements of these matrices are the coefficients that appear in the following expansions:

$$a_1(\theta) = \sum_{s=0}^{\infty} a^s_1 P^s_{00}(\cos\theta), \tag{15}$$

$$a_2(\theta) + a_3(\theta) = \sum_{s=2}^{\infty} (a^s_2 + a^s_3) P^s_{22}(\cos\theta), \tag{16}$$

$$a_2(\theta) - a_3(\theta) = \sum_{s=2}^{\infty} (a^s_2 - a^s_3) P^2_{2-2}(\cos\theta), \tag{17}$$

$$a_4(\theta) = \sum_{s=0}^{\infty} a^s_4 P^s_{00}(\cos\theta), \tag{18}$$

$$b_1(\theta) = \sum_{s=2}^{\infty} b^s_1 P^s_{02}(\cos\theta), \tag{19}$$

$$b_2(\theta) = \sum_{s=2}^{\infty} b^s_2 P^s_{02}(\cos\theta), \tag{20}$$

where θ is the scattering angle, and the functions $a_1(\theta)$ to $b_2(\theta)$ are the elements of the scattering matrix:[5]

$$\mathbf{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}. \tag{21}$$

For an ensemble of identical, randomly-oriented, axially-symmetric particles, the expansion coefficients a^s_1 to b^s_2 are given by

$$a^s_1 = g^s_{00} + g^s_{0\text{-}0}, \tag{22}$$

$$a^s_2 = g^s_{22} + g^s_{2\text{-}2}, \tag{23}$$

$$a^s_3 = g^s_{22} - g^s_{2\text{-}2}, \tag{24}$$

$$a^s_4 = g^s_{00} - g^s_{0\text{-}0}, \tag{25}$$

$$b_1^s = 2\,\mathrm{Re}\, g_{02}^s, \tag{26}$$

$$b_2^s = 2\,\mathrm{Im}\, g_{02}^s, \tag{27}$$

where[16]

$$g_{00}^s = \sum_{n=1}^{\infty} \sum_{\hat{n}=\hat{n}_{\min}}^{n+s} h_{sn\hat{n}} C_{n1\,s0}^{\hat{n}1} \sum_{m=-M}^{M} C_{nm\,s0}^{\hat{n}m} D_{mn\hat{n}}^{00}, \tag{28}$$

$$g_{0\text{-}0}^s = \sum_{n=1}^{\infty} \sum_{\hat{n}=\hat{n}_{\min}}^{n+s} h_{sn\hat{n}} (-1)^{n+\hat{n}+s} C_{n1\,s0}^{\hat{n}1} \sum_{m=-M}^{M} C_{nm\,s0}^{\hat{n}m} D_{mn\hat{n}}^{0\text{-}0}, \tag{29}$$

$$g_{22}^s = \sum_{n=1}^{\infty} \sum_{\hat{n}=\hat{n}_{\min}}^{n+s} h_{sn\hat{n}} C_{n-1\,s2}^{\hat{n}1} \sum_{m=m_{\min}}^{m_{\max}} C_{n-m\,s2}^{\hat{n}2-m} D_{mn\hat{n}}^{22}, \tag{30}$$

$$g_{2\text{-}2}^s = \sum_{n=1}^{\infty} \sum_{\hat{n}=\hat{n}_{\min}}^{n+s} h_{sn\hat{n}} (-1)^{n+\hat{n}+s} C_{n-1\,s2}^{n1} \sum_{m=m_{\min}}^{m_{\max}} C_{n-m\,s2}^{\hat{n}2-m} D_{mn\hat{n}}^{2\text{-}2}, \tag{31}$$

$$g_{02}^s = -\sum_{n=1}^{\infty} \sum_{\hat{n}=\hat{n}_{\min}}^{n+s} h_{sn\hat{n}} C_{n1\,s0}^{\hat{n}1} \sum_{m=m_{\min}}^{m_{\max}} C_{n-m\,s2}^{\hat{n}2-m} D_{mn\hat{n}}^{02}. \tag{32}$$

Here,

$$h_{sn\hat{n}} = \frac{(2s+1)\pi}{k^2 C_{\mathrm{sca}}} \left[\frac{2n+1}{2\hat{n}+1}\right]^{1/2}, \tag{33}$$

$$D_{mn\hat{n}}^{00} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B_{mnn_1}^1 [B_{m\hat{n}n_1}^1]^*, \tag{34}$$

$$D_{mn\hat{n}}^{0\text{-}0} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B_{mnn_1}^2 [B_{m\hat{n}n_1}^2]^*, \tag{35}$$

$$D_{mn\hat{n}}^{22} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B_{mnn_1}^1 [B_{2-m\hat{n}n_1}^1]^*, \tag{36}$$

$$D_{mn\hat{n}}^{2\text{-}2} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B_{mnn_1}^2 [B_{2-m\hat{n}n_1}^2]^*. \tag{37}$$

$$D_{mn\hat{n}}^{02} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B_{mnn_1}^2 [B_{2-m\hat{n}n_1}^1]^*, \tag{38}$$

$$B_{mnn_1}^j = \sum_{n'=\max(1,|n-n_1|)}^{n+n_1} C_{nm\,n_1 1-m}^{n'1} A_{nn'n_1}^j, \quad j=1,2, \tag{39}$$

$$A_{nn'n_1}^j = \frac{i^{n'-n}}{\sqrt{2n'+1}} \sum_{m_1=-M_1}^{M_1} C_{nm_1 n_1 0}^{n'm_1} T_{m_1 nn'}^j, \quad j=1,2, \tag{40}$$

$$T_{mnn'}^1 = \hat{T}_{mnmn'}^{11} + \hat{T}_{mnmn'}^{12} + \hat{T}_{mnmn'}^{21} + \hat{T}_{mnmn'}^{22}, \tag{41}$$

$$T_{mnn'}^2 = \hat{T}_{mnmn'}^{11} + \hat{T}_{mnmn'}^{12} - \hat{T}_{mnmn'}^{21} - \hat{T}_{mnmn'}^{22}, \tag{42}$$

$\hat{n}_{\min} = \max(1, |n-s|)$, $M = \min(n, \hat{n})$, $m_{\min} = \max(-n, -\hat{n}+2)$, $m_{\max} = \min(n, \hat{n}+2)$, $M_1 = \min(n, n')$, k is a wavenumber, and $C_{n_1 m_1 n_2 m_2}^{nm}$ are Clebsch–Gordan coefficients related to Wigner $3j$-symbols by

$$C_{n_1 m_1 n_2 m_2}^{nm} = (-1)^{n_1+n_2+m} \sqrt{2n+1} \begin{pmatrix} n_1 & n_2 & n \\ m_1 & m_2 & -m \end{pmatrix}. \tag{43}$$

In Eqs. (41) and (42), the quantities $\hat{T}_{mnm'n'}^{ij} = \delta_{mm'} \hat{T}_{mnmn'}^{ij}$ are elements of the T-matrix of the axially-symmetric particle calculated in the spherical coordinate system with the z-axis along the

axis of particle symmetry.[21] The scattering cross section averaged over the uniform distribution of particle orientations is given by[36]

$$C_{\text{sca}} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \sum_{m=0}^{\min(n,n')} \sum_{i,j=1,2} (2-\delta_{m0}) \, |\hat{T}^{ij}_{mnmn'}|^2. \tag{44}$$

The single scattering albedo w is given by

$$w = C_{\text{sca}}/C_{\text{ext}}, \tag{45}$$

where the orientationally-averaged extinction cross section is given by[37]

$$C_{\text{ext}} = -\frac{2\pi}{k^2} \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (2-\delta_{m0})(\hat{T}^{11}_{mnmn} + \hat{T}^{22}_{mnmn}). \tag{46}$$

In our computer calculations, we used the following scheme to determine the Stokes parameters of the reflected light. First, the expansion coefficients a_1^s to b_2^s and the single scattering albedo w were computed by using Eqs. (22)–(46); for details of these computations, see Ref. 16. Next, the matrices $\mathbf{Z}^m(u, u')$ were computed [see Eqs. (11)–(14)], and the matrices $\mathbf{R}^m(t; \mu, \mu_0)$ were determined by using the fast invariant imbedding method of Sato et al,[38] as modified in Ref. 39, to solve Eq. (9) numerically. Finally, Eqs. (2)–(8) were used.

Our computer programs were tested by comparing our computations with those reported in Refs. 31, 32, 40, and 41. De Haan[40] has computed the expansion coefficients a_1^s to b_2^s for randomly-oriented, identical, homogeneous spheroids by using the method of numerical angle integrations.[13] De Rooij,[31] Hovenier and de Haan,[41] and de Haan et al[32] have computed the Stokes parameters for reflected light for finite, plane-parallel slabs composed of Deirmendjian's[42] water-haze L by using the layer separation[31] and adding[32] methods. Also, general inequalities,[6] which must be satisfied by the expansion coefficients a_1^s to b_2^s, were used for checking purposes. In all of the cases considered, excellent agreement was found.

3. NUMERICAL RESULTS

The problem of computing light scattering by representative samples of particle shapes, sizes, and refractive indices is rather complex even if only the single scattering of light is considered, and becomes much more complicated if the multiple-scattering processes are taken into account. Therefore, in this paper, we do not aim to extensively study how the size, morphology, and composition of the scattering particles influence the Stokes parameters of the reflected light. On the other hand, scientific literature still lacks benchmark results for polarized light multiply-scattered by nonspherical particles. To provide others with accurate numbers, that may be used for checking purposes, we tabulate in this section some numerical results. More specifically, we use the computational scheme described to give detailed single- and multiple-scattering specifications of two simple particle models, which include homogeneous spheroids and Chebyshev particles.[23]

The shape of a spheroid in its natural spherical coordinate system is governed by the equation

$$r(\theta, \varphi) = a(\sin^2\theta + d^2\cos^2\theta)^{-1/2}, \quad d = a/b, \tag{47}$$

where b is the rotational semi-axis, and a is the horizontal semi-axis of the spheroid. The surface of a Chebyshev particle is governed by the equation[23]

$$r(\theta, \varphi) = r_0(1 + E\cos n\theta). \tag{48}$$

Computations in Tables 1–7 are given for two models, which are defined as follows.

(i) Model 1—prolate, randomly-oriented spheroids with a refractive index $m_r = 1.5 + 0.02i$, $d = 1/2$, and a size parameter $kr_{\text{ev}} = 5$, where r_{ev} is the radius of the equal-volume sphere.

(ii) Model 2[25]—randomly-oriented Chebyshev particles with $m_r = 1.5 + 0.02i$, $n = 3$, $E = 0.1$, and $kr_{\text{ev}} = 5$.

Table 1. Expansion coefficients for Model 1.

s	a_1^s	a_2^s	a_3^s	a_4^s	b_1^s	b_2^s
0	1.000000	0.0	0.0	0.940723	0.0	0.0
1	2.361493	0.0	0.0	2.365327	0.0	0.0
2	3.210177	4.122741	4.022476	3.179396	-0.059241	0.068084
3	3.678993	4.168309	4.134096	3.684857	-0.101014	-0.031627
4	3.803881	4.217701	4.151467	3.773201	-0.112180	-0.082074
5	3.626437	4.022923	4.014706	3.645650	-0.134336	-0.095261
6	3.307868	3.615506	3.574150	3.312758	-0.181818	-0.175668
7	2.772600	3.069676	3.037392	2.782695	-0.126610	-0.250945
8	2.131668	2.386923	2.341988	2.119656	-0.081813	-0.282060
9	1.470945	1.642812	1.614871	1.460541	0.013722	-0.217659
10	0.945403	1.058590	1.043341	0.940195	0.026799	-0.138290
11	0.570641	0.639444	0.632411	0.570372	0.025400	-0.095059
12	0.316681	0.356714	0.350254	0.315309	0.027475	-0.066634
13	0.150670	0.172949	0.168338	0.149405	0.022731	-0.036681
14	0.066050	0.076009	0.072291	0.064235	0.015125	-0.018570
15	0.022833	0.026653	0.024793	0.021814	0.007765	-0.006356
16	0.007569	0.008823	0.007956	0.007029	0.003040	-0.002095
17	0.001991	0.002338	0.002030	0.001786	0.001020	-0.000511
18	0.000497	0.000582	0.000487	0.000430	0.000279	-0.000116
19	0.000103	0.000121	0.000096	0.000085	0.000067	-0.000021
20	0.000020	0.000023	0.000018	0.000016	0.000014	-0.000004
21	0.000003	0.000004	0.000003	0.000002	0.000002	-0.000001
22	0.000001	0.000001	0.000000	0.000000	0.000000	-0.000000
23	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000000

Computed values of the expansion coefficients a_1^s to b_2^s are given in Tables 1 and 2. In Tables 3 and 4, these expansion coefficients are used to compute the elements of the corresponding scattering matrices for a number of scattering angles. In Table 5, we give computed values of the efficiency factors for extinction Q_{ext}, scattering Q_{sca}, and absorption Q_{abs}, as well as the single scattering albedo w and asymmetry parameter of the phase function $\langle\cos\theta\rangle$. Here,

$$Q_{\text{ext}} = C_{\text{ext}}/S, \tag{49}$$

$$Q_{\text{sca}} = C_{\text{sca}}/S, \tag{50}$$

$$Q_{\text{abs}} = Q_{\text{ext}} - Q_{\text{sca}}, \tag{51}$$

and

$$\langle\cos\theta\rangle = \frac{1}{2}\int_{-1}^{+1} \mathrm{d}(\cos\theta)a_1(\theta)\cos\theta = a_1^1/3, \tag{52}$$

where $S = \pi r_{\text{ev}}^2$ is the geometrical cross section of the equal-volume sphere.

Table 2. As in Table 1, for Model 2.

s	a_1^s	a_2^s	a_3^s	a_4^s	b_1^s	b_2^s
0	1.000000	0.0	0.0	0.939308	0.0	0.0
1	2.236333	0.0	0.0	2.262514	0.0	0.0
2	3.036196	4.053366	3.923276	2.974229	-0.119692	0.068848
3	3.273077	3.708191	3.720823	3.310402	-0.138266	-0.025325
4	3.404603	3.878463	3.795519	3.369212	-0.146049	-0.012113
5	3.356898	3.558753	3.566775	3.413946	-0.205064	-0.092534
6	3.138738	3.521084	3.418616	3.078227	-0.217630	-0.124859
7	2.755695	2.891310	2.978509	2.910137	-0.260905	-0.164285
8	2.076921	2.514331	2.391581	2.058262	-0.380484	-0.462129
9	1.315215	1.442898	1.398554	1.336372	-0.055886	-0.408714
10	0.705588	0.915850	0.831917	0.661042	-0.016911	-0.331468
11	0.163833	0.209445	0.188341	0.152923	0.063825	-0.095242
12	0.038517	0.049089	0.041786	0.034161	0.020916	-0.017810
13	0.007243	0.009215	0.007429	0.006081	0.004822	-0.002500
14	0.001124	0.001423	0.001069	0.000881	0.000857	-0.000270
15	0.000148	0.000186	0.000128	0.000107	0.000124	-0.000023
16	0.000017	0.000021	0.000013	0.000011	0.000015	-0.000002
17	0.000002	0.000002	0.000001	0.000001	0.000002	-0.000000
18	0.000000	0.000000	0.000000	0.000000	0.000000	-0.000000

Table 3. Elements of the scattering matrix for Model 1.

θ	a_1	a_2	a_3	a_4	b_1	b_2
0°	29.4465	29.3910	29.3910	29.3355	0.0	0.0
15°	15.3434	15.3019	15.2892	15.2614	0.1174	0.5475
30°	2.3881	2.3655	2.3060	2.3085	0.3391	0.2771
45°	0.7646	0.7477	0.6890	0.6985	0.1085	-0.1000
60°	0.4811	0.4591	0.4251	0.4439	0.0044	-0.0326
75°	0.2362	0.1977	0.1506	0.1871	0.0472	-0.0155
90°	0.1969	0.1353	0.0788	0.1379	0.0244	-0.0734
105°	0.1644	0.0782	0.0292	0.1129	0.0006	-0.0499
120°	0.1243	0.0203	-0.0249	0.0750	-0.0059	-0.0327
135°	0.0917	0.0173	-0.0287	0.0397	-0.0124	-0.0132
150°	0.0779	0.0376	-0.0325	0.0019	0.0205	0.0036
165°	0.0992	0.0643	-0.0609	-0.0286	0.0201	0.0057
180°	0.1331	0.0813	-0.0813	-0.0296	0.0	0.0

Table 4. As in Table 3, for Model 2.

θ	a_1	a_2	a_3	a_4	b_1	b_2
0°	26.5101	26.5090	26.5090	26.5078	0.0	0.0
15°	14.5850	14.5798	14.5584	14.5543	0.3718	0.6911
30°	2.0048	1.9950	1.8911	1.8851	0.5536	0.2526
45°	1.0499	1.0442	1.0072	1.0050	0.0368	-0.2639
60°	0.6361	0.6335	0.6110	0.6124	0.0717	0.1032
75°	0.2641	0.2607	0.1889	0.1894	0.1237	-0.0860
90°	0.1805	0.1776	0.1601	0.1614	-0.0115	-0.0568
105°	0.1597	0.1553	0.1315	0.1348	0.0518	0.0011
120°	0.1121	0.1088	0.0276	0.0288	0.0552	-0.0389
135°	0.1527	0.1460	0.0672	0.0727	-0.0656	-0.0931
150°	0.2066	0.2006	0.1786	0.1818	0.0190	0.0070
165°	0.1793	0.1757	-0.1063	-0.1053	0.0999	0.0769
180°	0.2933	0.2852	-0.2852	-0.2771	0.0	0.0

Table 5. Computed values of the efficiency factors for extinction Q_{ext}, scattering Q_{sca}, and absorption Q_{abs}, as well as the single scattering albedo w and asymmetry parameter of the phase function $\langle \cos\theta \rangle$.

	Q_{ext}	Q_{sca}	Q_{abs}	w	<cosθ>
Model 1	3.94029	3.50127	0.439016	0.888583	0.787164
Model 2	3.68507	3.22742	0.457646	0.875811	0.745444

In Tables 6 and 7, we give computed values of the Stokes parameters I, Q, U, and V of the reflected light for homogeneous slabs of optical thickness $t = 2$. The computations are presented for $\mathbf{R}^m(\mu, \mu_0) = \mathbf{0}$, where $\mathbf{0}$ is a (4×4) zero matrix [see Eq. (10)], $\mathbf{F}_{in} = (1, 1, 0, 0)^T$, $\mu_0 = 1$, $\varphi_0 = 0°$, and $\varphi = 0$, 22.5 and 30°. It should be noted that for $\mu_0 = 1$, only the components $\mathbf{R}^0(t; \mu, \mu_0)$ and $\mathbf{R}^2(t; \mu, \mu_0)$ contribute to the Stokes parameters of the reflected light.[41] Also, for $\varphi = 0°$, the Stokes parameters U and V are zero for any μ.[41] In solving the invariant imbedding equation numerically, a Gaussian quadrature formula with $n_* = 20$ division points was used. This number of division points was chosen to ensure the accuracy of the numbers in Tables 6 and 7 within ± 1 in the last digits given.

In our computations, a computer ES 1061 was used. The CPU time for computing the expansion coefficients was 2 min for Model 1 and 4 min for Model 2. Computation of the Stokes parameters of the reflected light required approx. 6 min for both models.

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Table 6. Stokes parameters of reflected light for Model 1 with $t = 2$, $\mathbf{R}^m(\mu, \mu_0) = \mathbf{0}$, $\mathbf{F}_{in} = (1, 1, 0, 0)^T$, $\mu_0 = 1$, $\varphi_0 = 0°$, and $\varphi = 0$, 22.5, and 30°.

μ	I			Q			U		V	
	0°	22.5°	30°	0°	22.5°	30°	22.5°	30°	22.5°	30°
0.05	0.11798	0.11618	0.11490	0.05348	0.03962	0.02982	-0.01666	-0.02040	-0.01794	-0.02197
0.1	0.12641	0.12486	0.12377	0.05293	0.03894	0.02905	-0.01615	-0.01978	-0.01797	-0.02201
0.2	0.13178	0.13059	0.12976	0.04668	0.03410	0.02521	-0.01223	-0.01497	-0.01616	-0.01979
0.3	0.12733	0.12633	0.12562	0.03731	0.02727	0.02017	-0.00631	-0.00773	-0.01357	-0.01662
0.4	0.11673	0.11598	0.11544	0.02749	0.02008	0.01484	-0.00046	-0.00056	-0.01120	-0.01371
0.5	0.10270	0.10239	0.10217	0.01907	0.01371	0.00991	0.00388	0.00475	-0.00909	-0.01113
0.6	0.08761	0.08776	0.08787	0.01340	0.00926	0.00634	0.00632	0.00774	-0.00681	-0.00834
0.7	0.07408	0.07430	0.07445	0.01150	0.00789	0.00534	0.00745	0.00912	-0.00420	-0.00514
0.8	0.06484	0.06444	0.06415	0.01390	0.01023	0.00763	0.00821	0.01006	-0.00178	-0.00218
0.9	0.05974	0.05858	0.05777	0.01873	0.01439	0.01132	0.00970	0.01188	-0.00019	-0.00024
1.0	0.05622	0.05622	0.05622	0.01993	0.01409	0.00997	0.01409	0.01726	0.00000	0.00000

Table 7. As in Table 6, for Model 2.

μ	I			Q			U		V	
	0°	22°5	30°	0°	22°5	30°	22°5	30°	22°5	30°
0.05	0.10386	0.10302	0.10243	0.06284	0.04557	0.03336	-0.03406	-0.04172	-0.00856	-0.01048
0.1	0.11556	0.11366	0.11232	0.06713	0.04960	0.03722	-0.03396	-0.04160	-0.00643	-0.00787
0.2	0.13436	0.13000	0.12692	0.07569	0.05802	0.04552	-0.03184	-0.03900	-0.00451	-0.00552
0.3	0.14320	0.13700	0.13261	0.07955	0.06250	0.05045	-0.02665	-0.03264	-0.00509	-0.00623
0.4	0.13718	0.13069	0.12610	0.07288	0.05802	0.04751	-0.01730	-0.02118	-0.00670	-0.00820
0.5	0.11871	0.11388	0.11047	0.05657	0.04479	0.03645	-0.00631	-0.00773	-0.00887	-0.01087
0.6	0.09802	0.09643	0.09530	0.03917	0.02923	0.02220	-0.00011	-0.00013	-0.01158	-0.01419
0.7	0.08672	0.08819	0.08923	0.03102	0.02040	0.01289	-0.00338	-0.00414	-0.01292	-0.01583
0.8	0.08817	0.08957	0.09056	0.03539	0.02357	0.01521	-0.01074	-0.01315	-0.00836	-0.01024
0.9	0.09247	0.08961	0.08759	0.04401	0.03395	0.02683	-0.00330	-0.00404	0.00322	0.00394
1.0	0.08868	0.08868	0.08868	0.04941	0.03494	0.02471	0.03494	0.04279	0.00000	0.00000

REFERENCES

1. I. Kuščer and M. Ribarič, *Optica Acta* **6,** 42 (1959).
2. M. Herman, *C. R. Acad. Sci.* **260,** 468 (1965).
3. H. Domke, *Astrophys. Space Sci.* **29,** 379 (1974).
4. C. E. Siewert, *Astrophys. J.* **245,** 1080 (1981).
5. J. W. Hovenier and C. V. M. van der Mee, *Astron. Astrophys.* **128,** 1 (1983).
6. C. V. M. van der Mee and J. W. Hovenier, *Astron. Astrophys.* **228,** 559 (1990).
7. H. Domke, *Z. Meteorologie* **25,** 357 (1975).
8. O. I. Bugaenko, *Izv. AN SSSR, Ser. Fiz. Atmos. Okeana* **12,** 603 (1976).
9. W. A. de Rooij and C. C. A. H. van der Stap, *Astron. Astrophys.* **131,** 237 (1984).
10. P. Vestrucci and C. E. Siewert, *JQSRT* **31,** 177 (1984).
11. V. M. Loskutov, *Vestnik Leningrad Gosud. Univ., Ser. 1* **3,** 95 (1987).
12. M. I. Mishchenko, *Kinem. Fiz. Nebes. Tel* **6,** No. 1, 91 (1990).
13. J. F. de Haan, "Effects of Aerosols on the Brightness and Polarization of Cloudless Planetary Atmospheres," Ph. D. Thesis, Free University, Amsterdam (1987).
14. S. Asano and G. Yamamoto, *Appl. Opt.* **14,** 29 (1975).
15. R. W. Schaefer, "Calculations of the Light Scattered by Randomly Oriented Ensembles of Spheroids of Size Comparable to the Wavelength," Ph. D. Thesis, State University of New York, Albany, NY (1980).
16. M. I. Mishchenko, "Light Scattering by Randomly Oriented Nonspherical Particles," Preprint 90-43E, Institute for Theoretical Physics, Acad. Sci. Ukrainian SSR, Kiev (1990).
17. P. Waterman, *Proc. IEEE* **53,** 805 (1965).
18. P. C. Waterman, *Phys. Rev. D* **3,** 825 (1971).
19. P. Barber and C. Yeh, *Appl. Opt.* **14,** 2864 (1975).
20. S. Ström, *Am. J. Phys.* **43,** 1060 (1975).
21. L. Tsang, J. A. Kong, and R. T. Shin, *Radio Sci.* **19,** 629 (1984).
22. D.-S. Wang and P. W. Barber, *Appl. Opt.* **18,** 1190 (1979).
23. A. Mugnai and W. J. Wiscombe, *J. Atmos. Sci.* **37,** 1291 (1980).
24. A. Lakhtakia, V. K. Varadan, and V. V. Varadan, *Appl. Opt.* **23,** 3502 (1984).
25. W. J. Wiscombe and A. Mugnai, "Single Scattering from Nonspherical Chebyshev Particles: a Compendium of Calculations," NASA Ref. Publ. 1157, NASA/GSFC, Greenbelt, MD (1986).
26. S. Ström and W. Zheng, *IEEE Trans. Antennas Propag.* **AP-36,** 376 (1988).
27. W. Zheng, *IEEE Trans. Antennas Propag.* **AP-36,** 1396 (1988).
28. W. Zheng and S. Ström, *IEEE Trans. Antennas Propag.* **AP-37,** 373 (1989).
29. A. Ishimaru, C. Yeh, and D. Lesselier, *Appl. Opt.* **23,** 4132 (1984).
30. A. Ishimaru, D. Lesselier, and C. Yeh, *Radio Sci.* **19,** 1356 (1984).
31. W. A. de Rooij, "Reflection and Transmission of Polarized Light by Planetary Atmospheres," Ph. D. Thesis, Free University, Amsterdam (1985).
32. J. F. de Haan, P. B. Bosma, and J. W. Hovenier, *Astron. Astrophys.* **183,** 371 (1987).
33. V. A. Ambartsumian, *Astron. Zh.* **19,** 30 (1942).
34. S. Chandrasekhar, *Radiative Transfer*, Oxford Univ. Press, London (1950).
35. I. M. Gel'fand, R.A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications*, Pergamon Press, Oxford (1963).
36. M. I. Mishchenko, *Kinem. Fiz. Nebes, Tel.* **6,** No. 5, 95 (1990).
37. M. I. Mishchenko, *Astrophys. Space Sci.* **164,** 1 (1990).
38. M. Sato, K. Kawabata, and J. E. Hansen, *Astrophys. J.* **216,** 947 (1977).
39. M. I. Mishchenko, *JQSRT* **43,** 163 (1990).
40. J. F. de Haan, private communication (1990).
41. J. W. Hovenier and J. F. de Haan, *Astron. Astrophys.* **146,** 185 (1985).
42. D. Deirmendjian, *Electromagnetic Scattering on Spherical Polydispersions*, Elsevier, New York, NY (1969).