RADIATION PRESSURE ON RANDOMLY-ORIENTED NONSPHERICAL PARTICLES

(Letter to the Editor)

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Abstract. We describe an efficient method to compute the efficiency factors for radiation pressure for randomly-oriented axially-symmetric nonspherical grains. The method is based on Waterman's T-matrix approach and may be used in computations for grains with sizes smaller than or of the order of the wavelength. An illustrative numerical example for randomly oriented spheroidal grains is given.

1. Introduction

In solving many astrophysical problems, one has to take into account radiation pressure upon small particles (see, e.g., Mathews, 1967; Kwok, 1975; Tarafdar and Wickramasinghe, 1976; Martin, 1978; Burns *et al.*, 1979; Simpson *et al.*, 1980; Voshchinnikov and Il'in, 1983; Jura, 1984; Voshchinnikov, 1986; Fröhlich and Notni, 1988; Gustafson, 1989). Though in most cases of astrophysical interest the scattering particles are nonspherical, in practical computations they are usually replaced by 'equivalent' spheres, and then the Mie theory is employed, i.e., the following formula is used (van de Hulst, 1957):

$$C_{pr} = \frac{2\pi}{k^2} \operatorname{Re} \sum_{n=1}^{\infty} \left\{ (2n+1)(a_n + b_n) + \frac{2n(n+2)}{n+1} (a_n a_{n+1}^* + b_n b_{n+1}^*) + \right.$$

$$\left. + \frac{2(2n+1)}{n(n+1)} a_n b_n^* \right\}, \tag{1}$$

where C_{pr} is the cross-section of radiation pressure, a_n and b_n are Mie coefficients, $k = 2\pi/\lambda$, λ is the wavelength, and the asterisk denotes the complex conjugate values. The purpose of the present paper is to discuss a more general problem: namely, to consider computation of the cross-sections of radiation pressure for ensembles of randomly-oriented axially-symmetric nonspherical particles. In Section 2, we use Waterman's (1971) T-matrix approach to obtain some generalization of Equation (1), which is applicable to this more general case, and briefly describe the corresponding computational scheme. In Section 3, some illustrative numerical results for spheroidal grains are given.

It is worthwhile to note here that Onaka (1980) and, recently, Voshchinnikov (1990) considered computation of the radiation pressure upon homogeneous and layered

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spheroidal particles. Nevertheless, their results are of limited use because they considered only particles in fixed orientation with respect to the direction of light propagation, whereas in nature nonspherical grains are most frequently distributed over orientations rather than perfectly aligned.

2. Computational Method

The cross-section of radiation pressure for an ensemble of randomly-oriented identical nonspherical particles is given by (e.g., van de Hulst, 1957; Bohren and Huffman, 1983)

$$C_{pr} = C_{\text{ext}} - C_{\text{sca}} \langle \cos\theta \rangle \,, \tag{2}$$

where C_{ext} is the extinction cross-section, C_{sca} is the scattering cross-section, and $\langle \cos\theta \rangle$ is the average cosine of the scattering angle (or asymmetry parameter of the phase function). All these quantities are assumed to be averaged over the uniform orientational distribution of a nonspherical particle. To compute these quantities, we shall use the T-matrix approach (Waterman, 1971), which seems to be the most efficient tool of solving light scattering problems for homogeneous and layered axially-symmetric nonspherical particles of size not too large as compared with the wavelength.

Let us consider a nonspherical particle with a fixed orientation with respect to the laboratory reference frame. Using the T-matrix approach, we expand the fields that are incident upon (superscript i) and scattered by (superscript s) this particle in vector spherical waves as follows (Tsang *et al.*, 1984):

$$\mathbf{E}^i(\mathbf{r}) = \sum_{mn} [a_{mn}\, \mathrm{Rg}\mathbf{M}_{mn}(k\mathbf{r}) + b_{mn}\, \mathrm{Rg}\mathbf{N}_{mn}(k\mathbf{r})] \tag{3}$$

and

$$\mathbf{E}^s(\mathbf{r}) = \sum_{mn} [\, p_{mn}\mathbf{M}_{mn}(k\mathbf{r}) + q_{mn}\mathbf{N}_{mn}(k\mathbf{r})] \,. \tag{4}$$

The vector spherical waves $\mathrm{Rg}\mathbf{M}_{mn}$ and $\mathrm{Rg}\mathbf{N}_{nm}$ in Equation (3) have a Bessel function radial dependence, while the functions $\mathbf{M}_{mn}$ and $\mathbf{N}_{mn}$ in Equation (4) have a Hankel function radial dependence. The expansion coefficients of the incident field a_{mn} and b_{mn} are assumed to be known (e.g., for a plane incident wave they are given by simple analytical expressions), whereas the expansion coefficients of the scattered field p_{mn} and q_{mn} are initially unknown. The relation between these coefficients is linear and is given by a transition matrix (or T-matrix) $\mathbf{T}$ as

$$p_{mn} = \sum_{m'n'} [T^{11}_{mnm'n'} a_{m'n'} + T^{12}_{mnm'n'} b_{m'n'}] \,, \tag{5}$$

$$q_{mn} = \sum_{m'n'} [T^{21}_{mnm'n'} a_{m'n'} + T^{22}_{mnm'n'} b_{m'n'}] \,. \tag{6}$$

The elements of the T-matrix do not depend upon the directions of propagation and states of polarization of the incident and scattered fields. They depend only upon the size, morphology, and composition of the scattering particle, as well as upon its

orientation with respect to the laboratory reference frame. The T-matrix can be computed by using special formulae, and then one easily computes the expansion coefficients of the scattered field and the scattered field itself (see Equations (5)–(6) and (4), respectively), as well as the amplitude scattering matrix, optical cross-sections, single scattering albedo, etc.

The T-matrix approach is especially efficient for axially-symmetric scatterers, because in this case the T-matrix can be diagonalized with respect to the indices m and m' by computing it in the natural (or body) coordinate system with the z-axis along the axis of particle symmetry (see, e.g., Tsang *et al.*, 1984)

$$\hat{T}^{ij}_{mnm'n'} = \delta_{mm'} \hat{T}^{ij}_{mnmn'} , \quad i, j = 1, 2 ; \tag{7}$$

where $\delta_{mm'}$ is the Kronecker delta, and $\hat{\mathbf{T}}$ is the T-matrix of the axially-symmetric particle computed in its natural coordinate system. After computing the matrix $\hat{\mathbf{T}}$, the T-matrix of the particle with respect to the laboratory reference frame can easily be computed by using the formula (Varadan, 1980; Tsang *et al.*, 1984)

$$\mathbf{T} = \mathbf{D}^{-1}(\alpha\beta\gamma)\hat{\mathbf{T}}\mathbf{D}(\alpha\beta\gamma) , \tag{8}$$

where $\mathbf{D}$ are Wigner's D-matrices (see, e.g., Varshalovich *et al.*, 1975), and the Eulerian angles of rotation α, β, and γ specify the orientation of the particle with respect to the laboratory reference frame.

An important advantage of the T-matrix approach is that it is ideally suited to calculate orientationally averaged quantities for randomly oriented or partially aligned nonspherical particles (Tsang *et al.*, 1984; Mishchenko, 1990a–e). In particular, in a recent paper we have shown (Mishchenko, 1990a) that the extinction cross-section, averaged over the uniform orientational distribution of identical axially-symmetric particles, is given by

$$C_{\text{ext}} = -\frac{2\pi}{k^2} \operatorname{Re} \sum_{n=1}^{\infty} \sum_{m=0}^{n} (2 - \delta_{m0}) (\hat{T}^{11}_{mnmn} + \hat{T}^{22}_{mnmn}) . \tag{9}$$

To compute the quantity $C_{\text{sca}} \langle \cos\theta \rangle$, we proceed as follows. By definition (see, e.g., van de Hulst, 1957),

$$\langle \cos\theta \rangle = \frac{1}{2} \int_{-1}^{+1} d(\cos\theta)\Phi(\theta) \cos\theta , \tag{10}$$

where $\Phi(\theta)$ is the phase function. Let us expand the function $\Phi(\theta)$ in Legendre polynomials as

$$\Phi(\theta) = \sum_{s=0}^{\infty} x_s P_s (\cos\theta) . \tag{11}$$

Then, by using Equation (11) and the orthogonality relation

$$\int_{-1}^{+1} \mathrm{d}u\, P_s(u) P_{s'}(u) = \frac{2}{2s+1}\, \delta_{ss'}\,, \tag{12}$$

we have

$$\langle \cos\theta \rangle = x_1/3\,. \tag{13}$$

Formulae for the expansion coefficients x_s for an ensemble of randomly oriented, identical, axially-symmetric particles have been derived in our recent paper (Mishchenko, 1990c). By using these formulae, one easily obtains

$$C_{\mathrm{sca}} \langle \cos\theta \rangle = \pi k^{-2} \sum_{n=1}^{\infty} \sum_{\hat{n}=\max(1,|n-1|)}^{n+1} \left[\frac{2n+1}{2\hat{n}+1}\right]^{1/2} \times$$
$$\times\, C^{\hat{n}1}_{n1\,10} \sum_{m=-M}^{M} C^{\hat{n}m}_{nm\,10} [D_{mn\hat{n}} - (-1)^{n+\hat{n}} \hat{D}_{mn\hat{n}}]\,, \tag{14}$$

where

$$M = \min(n, \hat{n}),$$

$$D_{mn\hat{n}} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B^1_{mnn_1} [B^1_{m\hat{n}n_1}]^*\,, \tag{15}$$

$$\hat{D}_{mn\hat{n}} = \sum_{n_1=|m-1|}^{\infty} (2n_1+1) B^2_{mnn_1} [B^2_{m\hat{n}n_1}]^*\,, \tag{16}$$

$$B^j_{mnn_1} = \sum_{n'=\max(1,|n-n_1|)}^{n+n_1} C^{n'1}_{nm\,n_1 1\,-m} A^j_{nn'n_1}\,, \qquad j = 1, 2\,, \tag{17}$$

$$A^j_{nn'n_1} = \frac{i^{n'-n}}{\sqrt{2n'+1}} \sum_{m_1=-M_1}^{M_1} C^{n'm_1}_{nm_1\,n_1 0} T^j_{m_1 nn'}\,, \qquad M_1 = \min(n, n')\,, \tag{18}$$

$$T^1_{mnn'} = \hat{T}^{11}_{mnmn'} + \hat{T}^{12}_{mnmn'} + \hat{T}^{21}_{mnmn'} + \hat{T}^{22}_{mnmn'}\,, \tag{19}$$

$$T^2_{mnn'} = \hat{T}^{11}_{mnmn'} + \hat{T}^{12}_{mnmn'} - \hat{T}^{21}_{mnmn'} - \hat{T}^{22}_{mnmn'}\,, \tag{20}$$

and $C^{nm}_{n_1 m_1\, n_2 m_2}$ are Clebsch–Gordan coefficients related to Wigner's $3j$-symbols by (e.g., Varshalovich *et al.*, 1975)

$$C^{nm}_{n_1 m_1\, n_2 m_2} = (-1)^{n_1+n_2+m} \sqrt{2n+1} \begin{pmatrix} n_1 & n_2 & n \\ m_1 & m_2 & -m \end{pmatrix}.$$

Thus, to compute the orientationally averaged cross-section of radiation pressure for an ensemble of randomly oriented, identical, axially-symmetric particles, it is sufficient to calculate the T-matrix with respect to the natural reference frame of the particle (i.e.,

the matrix $\hat{\mathbf{T}}$) and then to use Equations (19)–(20), (18), (17), (15)–(16), (14), (9), and (2). Formulae for computing the matrix $\hat{\mathbf{T}}$ for homogeneous axially-symmetric particles are given, e.g., by Tsang *et al.* (1984). Computation of the matrix $\hat{\mathbf{T}}$ for composite particles is considered by Peterson and Ström (1974), Ström and Zheng (1988), Zheng (1988), Zheng and Ström (1989), and Wang and Barber (1979). Numerical aspects of the T-matrix computations are extensively discussed by Wiscombe and Mugnai (1986). Finally, convenient formulae for computing the Clebsch–Gordan coefficients, appearing in Equations (14) and (17)–(18), are collected in Appendix B of Mishchenko (1990c).

It should be noted that for a spherical particle with spherically symmetric internal structure,

$$\hat{T}^{11}_{mnmn'} = -\delta_{nn'} b_n \,, \tag{21}$$

$$\hat{T}^{12}_{mnmn'} = \hat{T}^{21}_{mnmn'} = 0 \,, \tag{22}$$

$$\hat{T}^{22}_{mnmn'} = -\delta_{nn'} a_n \,, \tag{23}$$

where a_n and b_n are the Mie coefficients, if the particle is homogeneous, and their analogues, if the particle is radially inhomogeneous (see, e.g., Bohren and Huffman, 1983; Prishivalko *et al.*, 1984). By using the properties of the Clebsch–Gordan coefficients (e.g., Varshalovich *et al.*, 1975), one easily derives Equation (1) as a particular case of more general Equations (9) and (14).

3. Illustrative Numerical Results

In this section, we present some illustrative numerical results for randomly-oriented spheroidal grains. The shape of a spheroid in its natural reference frame is governed by the equation

$$r(\theta, \varphi) = a\left(\sin^2\theta + \frac{a^2}{b^2}\cos^2\theta\right)^{-1/2} , \tag{24}$$

where b is the rotational semi-axis, and a is the horizontal semi-axis of the spheroid. Another pair of parameters, that may be used to specify the shape of the spheroid, is (r_{ev}, d), where $d = a/b$ is the ratio of the semi-axes, and r_{ev} is the radius of the equal-volume sphere given by

$$r_{ev} = ad^{-1/3} \,. \tag{25}$$

In Table I we give numerical results for particles with $r_{ev} = 0.1$ μm and for two values of the refractive index: $m_r = 1.7 + 0.03i$ and $m_r = 2.5 + i$. Instead of the cross-sections, the efficiency factors for radiation pressure are displayed which are given by

$$Q_{pr} = C_{pr}/S \,, \tag{26}$$

where $S = \pi r_{ev}^2$ is the geometrical cross-section of the equal volume sphere. The computational parameters, that determine the accuracy of the T-matrix computations (see,

TABLE I

Efficiency factors for radiation pressure for spheres and randomly oriented spheroids with $r_{ev} = 0.1$ μm

Wavelength, μm	$m_r = 1.7 + 0.03i$			$m_r = 2.5 + i$		
	Prolate spheroids, $d = 1/2$	Spheres, $d = 1$	Oblate spheroids, $d = 2$	Prolate spheroids, $d = 1/2$	Spheres, $d = 1$	Oblate spheroids, $d = 2$
0.1	1.368	1.284	1.441	1.540	1.441	1.552
0.2	1.564	1.794	1.590	1.896	1.782	1.908
0.3	1.385	1.425	1.316	2.231	2.114	2.244
0.4	0.9292	0.9772	0.9687	2.540	2.394	2.528
0.5	0.6647	0.6535	0.6666	2.744	2.759	2.740
0.6	0.4612	0.4504	0.4609	2.746	2.828	2.788
0.7	0.3053	0.2959	0.3062	2.434	2.396	2.499
0.8	0.2060	0.1985	0.2072	2.009	1.852	2.047
0.9	0.1450	0.1391	0.1461	1.619	1.418	1.627
1.0	0.1068	0.1021	0.1076	1.309	1.110	1.303

e.g., Wiscombe and Mugnai, 1986), were chosen such that the accuracy of computing the efficiency factors for radiation pressure was better than 0.01%.

References

Bohren, C. F. and Huffman, D. R.: 1983, *Absorption and Scattering of Light by Small Particles*, John Wiley, New York.

Burns, J. A., Lamy, P. L., and Soter, S.: 1979, *Icarus* **40**, 1.

Fröhlich, H.-E. and Notni, P.: 1988, *Astron. Nachr.* **309**, 147.

Gustafson, B. Å. S.: 1989, *Astrophys. J.* **337**, 945.

Jura, M.: 1984, *Astrophys. J.* **282**, 200.

Kwok, S.: 1975, *Astrophys. J.* **198**, 583.

Martin, P. G.: 1978, *Cosmic Dust*, Oxford University Press, Oxford.

Mathews, W. G.: 1967, *Astrophys. J.* **147**, 965.

Mishchenko, M. I.: 1990a, *Astrophys. Space Sci.* **164**, 1.

Mishchenko, M. I.: 1990b, *Kinem. Fiz. Nebes. Tel* **6**, No. 5, 95.

Mishchenko, M. I.: 1990c, 'Light Scattering by Randomly Oriented Nonspherical Particles', Preprint No. ITP-90-43E, Institute for Theoretical Physics, Ukrainian Acad. Sci., Kiev (*J. Optical Soc. Amer. A*, accepted for publication).

Mishchenko, M. I.: 1990d, *Pis'ma Astron. Zh.* **16**, 951.

Mishchenko, M. I.: 1990e, *Astrophys. J.* (in press).

Onaka, T.: 1980, *Ann. Tokyo Astron. Obs.* **18**, 1.

Peterson, B. and Ström, S.: 1974, *Phys. Rev.* **D10**, 2670.

Prishivalko, A. P., Babenko, V. A., and Kuz'min, V. N.: 1984, *Scattering and Absorption of Light by Inhomogeneous and Anisotropic Spherical Particles*, Nauka i Tekhnika, Minsk (in Russian).

Simpson, I. C., Simons, S., and Williams, I. P.: 1980, *Astrophys. Space Sci.* **71**, 3.

Ström, S. and Zheng, W.: 1988, *IEEE Trans. Antennas Propag.* **AP-36**, 376.

Tarafdar, S. P. and Wickramasinghe, N. C.: 1976, *Astrophys. Space Sci.* **39**, 19.

Tsang, L., Kong, J. A., and Shin, R. T.: 1984, *Radio Sci.* **19**, 629.

van de Hulst, H. C.: 1957, *Light Scattering by Small Particles*, John Wiley, New York.

Varadan, V. K.: 1980, in V. K. Varadan and V. V. Varadan (eds.), *Acoustic, Electromagnetic and Elastic Wave Scattering – Focus on the T-Matrix Approach*, Pergamon Press, New York, p. 103.

Varshalovich, D. A., Moskalev, A. N., and Khersonskij, V. K.: 1975, *Quantum Theory of Angular Momentum*, Nauka, Leningrad (in Russian).
Voshchinnikov, N. V.: 1986, *Issled. Kosmich. Prostranstva* **25**, 98.
Voshchinnikov, N. V.: 1990, *Astron. Zh.* **67**, 851.
Voshchinnikov, N. V. and Il'in, V. B.: 1983, *Astron. Zh.* **60**, 1120.
Wang, D.-S. and Barber, P. W.: 1979, *Appl. Optics* **18**, 1190.
Waterman, P. C.: 1971, *Phys. Rev.* **D3**, 825.
Wiscombe, W. J. and Mugnai, A.: 1986, *Single Scattering from Nonspherical Chebyshev Particles: A Compendium of Calculations*, NASA Ref. Publ. 1157.
Zheng, W.: 1988, *IEEE Trans. Antennas Propag.* **AP-36**, 1396.
Zheng, W. and Ström, S.: 1989, *IEEE Trans. Antennas Propag.* **AP-37**, 373.