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MULTIPLE SCATTERING OF POLARIZED LIGHT IN ANISOTROPIC PLANE-PARALLEL MEDIA

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ABSTRACT

Transfer of polarized light in anisotropic plane-parallel media, consisting of arbitrarily oriented nonspherical particles, is considered. Adding equations and a complete set of invariant imbedding equations for reflection and transmission matrices are derived.

1. INTRODUCTION

In the present paper, we study the vector radiative transfer equation relevant to scattering media, consisting of randomly distributed particles. Unlike most of publications on this subject (see e.g. Refs. 1-21 and references therein), we do not use the common assumption that the scattering media are macroscopically isotropic. We only assume that the media are plane-parallel, and that scattering occurs without frequency redistribution.

The main purpose of our paper is to generalize several important equations derived previously for isotropic plane-parallel atmospheres. In the next

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section, we shall briefly discuss the main relationships relevant to the scattering of a plane electromagnetic wave by a single nonspherical particle and to the transfer of polarized light in a sparse, random distribution of such particles. Then, in Sec. 3, adding equations will be derived, which enable one to calculate reflection and transmission matrices of a combined slab provided that these quantities for each subslab are known. Also, calculation of the internal radiation field will be considered. In Sec. 4, the adding equations will be used to derive a complete set of invariant imbedding equations for the reflection and transmission matrices. Some consequences, valid for homogeneous layers, will be drawn. Finally, in Sec. 5, we shall briefly discuss the main results of our work.

Among previous publications on radiative transfer in anisotropic media, we note Refs. 22-31.

2. RADIATIVE TRANSFER EQUATION

2.1. Scattering of a plane electromagnetic wave by a single nonspherical particle

Consider a plane electromagnetic wave

$$\hat{E}^i(\hat{r}) = (E_1^i\,\hat{\theta}_i + E_2^i\,\hat{\varphi}_i)\exp(ik\hat{r}\hat{n}_i) \tag{1}$$

incident upon a nonspherical particle in a direction specified by the unit vector $\hat{n}_i = (u_i, \varphi_i)$. Here, $k = 2\pi/\lambda$, λ is a free space wavelength, $u_i = \cos\theta_i$, θ_i is a zenith angle, and φ_i is an azimuth angle; by subscripts 1 and 2 we label θ- and φ-components of the electric field, respectively, $\hat{\theta}_i$ and $\hat{\varphi}_i$ being the corresponding unit vectors. The origin of the spherical coordinate system is assumed to be inside the scattering particle. Note that $\hat{n}_i = \hat{\theta}_i \times \hat{\varphi}_i$. The time

factor $\exp(-i\omega t)$ is assumed and omitted throughout the paper. In the far-field zone $(kr \gg 1)$, the scattered wave becomes spherical and is given by (cf. Refs. 32 and 33)

$$\hat{E}^s(\hat{r}) = E_1^s(r,\hat{n}_s)\,\hat{\theta}_s + E_2^s(r,\hat{n}_s)\,\hat{\varphi}_s\ , \quad \hat{n}_s = \hat{r}/r\ ,$$

$$\hat{E}^s(\hat{r})\cdot\hat{r} = 0\ , \tag{2}$$

$$\begin{bmatrix} E_1^s \\ E_2^s \end{bmatrix} = \frac{e^{ikr}}{r}\,\hat{F}(\hat{n}_s,\hat{n}_i)\begin{bmatrix} E_1^i \\ E_2^i \end{bmatrix}\ , \quad \hat{F} = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix}\ ,$$

where $\hat{F}$ is an amplitude scattering matrix, which depends (besides $\hat{n}_i$ and $\hat{n}_s$) upon the size, shape, and refractive index of the scattering particle, as well as on its orientation with respect to the coordinate system.

In this paper, we use the Stokes parameters of the incident plane wave and the scattered spherical wave defined as (cf. Refs. 32 and 33)

$$\begin{aligned} I &= E_1E_1^* + E_2E_2^*\ , \\ Q &= E_1E_1^* - E_2E_2^*\ , \\ U &= E_1E_2^* + E_2E_1^*\ , \\ V &= i\,(E_1E_2^* - E_2E_1^*)\ , \end{aligned} \tag{3}$$

where the asterisc denotes complex conjugation. Also, the Stokes vector (or the intensity vector) is $\hat{I} = (I,Q,U,V)^T$, where "T" denotes matrix transposition. Writing the elements of the amplitude scattering matrix as $A_k = a_k \exp(ib_k)$, $k=1,\ldots,4$, and defining auxiliary quantities

$$M_k = A_k A_k^* = a_k^2 \quad ,$$

$$S_{kj} = \frac{1}{2}\,(A_k A_j^* + A_j A_k^*) = a_k a_j \cos(b_k - b_j) \quad , \tag{4}$$

$$D_{kj} = \frac{1}{2}\, i\,(A_k A_j^* - A_j A_k^*) = -\, a_k a_j \sin(b_k - b_j)$$

we have[32,33]

$$\hat{I}^s = \frac{1}{r^2}\,\hat{Z}(\hat{n}_s, \hat{n}_i)\,\hat{I}^i \quad , \tag{5}$$

where $\hat{Z}$ is the phase matrix with elements

$$Z_{11} = \frac{1}{2}\,(M_1 + M_2 + M_3 + M_4) \quad ,$$

$$Z_{12} = \frac{1}{2}\,(-M_1 + M_2 - M_3 + M_4) \quad ,$$

$$Z_{13} = S_{32} + S_{41} \quad ,$$

$$Z_{14} = D_{32} - D_{41} \quad ,$$

$$Z_{21} = \frac{1}{2}\,(-M_1 + M_2 + M_3 - M_4) \quad ,$$

$$Z_{22} = \frac{1}{2}\,(M_1 + M_2 - M_3 - M_4) \quad ,$$

$$Z_{23} = S_{32} - S_{41} \quad ,$$

$$Z_{24} = D_{32} + D_{41} \quad ,$$

$$Z_{31} = S_{42} + S_{31} \quad , \tag{6}$$

$$Z_{32} = S_{42} - S_{31} \quad ,$$

$$Z_{33} = S_{21} + S_{43} \quad ,$$

$$Z_{34} = -D_{21} - D_{43} \quad ,$$

$$Z_{41} = -D_{42} + D_{31} \quad ,$$

$$Z_{42} = -D_{42} - D_{31} \quad ,$$

$$Z_{43} = D_{21} - D_{43} \quad ,$$

$$Z_{44} = S_{21} - S_{43} \quad .$$

Through Eqs. (4) and (6), the elements of the phase matrix are expressed in 7 independent real variables a_k, $k=1,\ldots,4$, and b_k, $k=1,2,3$. Therefore, there must be 9 relations between the 16 elements of the phase matrix Z_{kj}, $k,j=1,\ldots,4$.[32] These relations are given in Ref.34(see also Refs. 35 and 36).

As was shown by Saxon,[37] the amplitude scattering matrix obeys the reciprocity relation

$$\hat{F}(\hat{n}_s,\hat{n}_i) = \hat{B}\,\hat{F}^T(-\hat{n}_i,-\hat{n}_s)\,\hat{B} \quad , \tag{7}$$

where $\hat{B} = \mathrm{diag}(1,-1)$. From Eqs. (4)-(7), we have

$$\hat{Z}(\hat{n}_s,\hat{n}_i) = \hat{P}\,\hat{Z}^T(-\hat{n}_i,-\hat{n}_s)\,\hat{P} \quad , \tag{8}$$

where $\hat{P} = \mathrm{diag}(1,1,-1,1)$.

For numerical calculation of the light scattering properties of a single nonspherical particle, several computational methods have been proposed. They are reviewed in Refs. 38-42.

2.2. Multiple scattering and the equation of transfer

The field $\hat{E}(\hat{r})$, multiply scattered by randomly distributed scatterers, is not a plane or a spherical wave. Furthermore, this field does not depend upon a "direction of propagation". Therefore, we can not insert directly the vector $\hat{E}(\hat{r})$ into Eqs. (3) to define the Stokes parameters, which must satisfy the scattering law Eq. (5). Preliminarly, the field $\hat{E}(\hat{r})$ has to be expanded in plane waves as follows:

$$\hat{E}(\hat{r}) = \int d\hat{n}\ \hat{e}(\hat{n})\ \exp(ik\hat{n}\hat{r}) \quad , \quad \hat{n}\cdot\hat{e}(\hat{n}) = 0 \quad , \tag{9}$$

where

$$\int d\hat{n} \ldots = \int_{-1}^{+1} du \int_{0}^{2\pi} d\varphi \ldots \quad .$$

For extensive discussion,we refer the reader to Refs. 43-49; here we cite only the final result, which is a consequence of the theory of multiple scattering of electromagnetic waves in sparse, discrete random media.

Quasi-homogeneous random electromagnetic field can be characterized by 4 real-valued functions $I(\hat{n})$, $Q(\hat{n})$, $U(\hat{n})$, and $V(\hat{n})$ of the same physical dimension, which may be interpreted as the Stokes parameters of a light beam propagating in the direction $\hat{n}$. The first Stokes parameter is the specific intensity of light and through the relation

$$dE = I(\hat{n})\ \hat{n}\ d\hat{s}\ dt\ do$$

is expressed in the amount of radiant energy which is transported during a time interval dt across an

element $d\hat{s}$ into a solid angle do about the direction $\hat{n}$. The intensity vector $\hat{I}(\hat{n}) = [I(\hat{n}),Q(\hat{n}),U(\hat{n}),V(\hat{n})]^T$ of radiation,multiply scattered in a sparse distribution of randomly positioned, independently scattering particles, obeys the vector radiative transfer equation

$$u \frac{d\,\hat{I}(z,\hat{n})}{dz} = n_p(z)\,\hat{K}(z,\hat{n})\,\hat{I}(z,\hat{n}) + n_p(z) \int d\hat{n}'\,\hat{Z}(z,\hat{n},\hat{n}')\,\hat{I}(z,\hat{n}') \quad , \qquad (10)$$

where the scattering medium is assumed to be plane-parallel, and the z-axis is directed towards the inward normal to the upper boundary of the medium. In Eq. (10), n_p is the number density of scattering particles, $\hat{Z}$ is the phase matrix, and $\hat{K}$ is the (4x4) extinction matrix, which is expressed in the elements of the amplitude scattering matrix as follows:

$$\hat{K}(\hat{n}) = \frac{2\pi}{k}\,\hat{k}(\hat{n}) \quad ,$$

$$k_{jj}(\hat{n}) = -\,\mathrm{Im}\,(A_1(\hat{n},\hat{n}) + A_2(\hat{n},\hat{n})) \;, \quad j=1,\ldots,4 \;,$$

$$k_{12}(\hat{n}) = k_{21}(\hat{n}) = \mathrm{Im}\,(A_1(\hat{n},\hat{n}) - A_2(\hat{n},\hat{n})) \quad ,$$

$$k_{13}(\hat{n}) = k_{31}(\hat{n}) = -\,\mathrm{Im}\,(A_4(\hat{n},\hat{n}) + A_3(\hat{n},\hat{n})) \quad ,$$

$$k_{14}(\hat{n}) = k_{41}(\hat{n}) = \mathrm{Re}\,(A_4(\hat{n},\hat{n}) - A_3(\hat{n},\hat{n})) \quad , \qquad (11)$$

$$k_{23}(\hat{n}) = -\,k_{32}(\hat{n}) = \mathrm{Im}\,(A_4(\hat{n},\hat{n}) - A_3(\hat{n},\hat{n})) \quad ,$$

$$k_{24}(\hat{n}) = -\,k_{42}(\hat{n}) = -\,\mathrm{Re}\,(A_4(\hat{n},\hat{n}) + A_3(\hat{n},\hat{n})) \quad ,$$

$$k_{34}(\hat{n}) = -\,k_{43}(\hat{n}) = \mathrm{Re}\,(A_2(\hat{n},\hat{n}) - A_1(\hat{n},\hat{n})) \quad .$$

The extinction matrix and the phase matrix, occurring in the equation of transfer, are averaged over the ensemble of noncorrelated, independent scatterers. In calculating the ensemble averages, the size, shape, refractive index, and orientation distributions of the scattering particles must be taken into account. The quadratic relations between the elements of the phase matrix of a single particle are, generally, lost in the averaging process. A selection of inequalities for the elements of the ensemble averaged phase matrix is given in Ref. 34.

Like the phase matrix of a single scatterer, the ensemble averaged phase matrix obeys the reciprocity relation Eq. (8). Using Eqs. (7) and (11), we easily derive the reciprocity relation for the extinction matrix:

$$\hat{K}(\hat{n}) = \hat{P}\ \hat{K}^T(-\hat{n})\ \hat{P} \quad . \tag{12}$$

The problem of a numerical calculation of the ensemble averaged extinction and phase matrices is, generally, very difficult. One of the methods for solving this problem, which is based on the Waterman's T-matrix approach,[50] was proposed by Tsang et al.[51]

3. THE ADDING EQUATIONS

In what follows, we consider a finite slab $[z_t;z_b]$, where by subscripts "t" and "b" the top and bottom values are labeled, respectively. Using the definition

$$t(z) = \int_{-\infty}^{z} dz'\ n_p(z') \tag{13}$$

we have

$$u \frac{d\,\hat{I}(t,\hat{n})}{dt} = \hat{K}(t,\hat{n})\,\hat{I}(t,\hat{n}) + \int d\hat{n}'\,\hat{Z}(t,\hat{n},\hat{n}')\,\hat{I}(t,\hat{n}') \;. \tag{14}$$

Let the (4x4) matrix $\hat{X}(t_1,t_2,\hat{n})$ be the solution of the equation

$$u \frac{d\,\hat{X}(t_1,t_2,\hat{n})}{dt_1} = \hat{K}(t_1,\hat{n})\,\hat{X}(t_1,t_2,\hat{n}) \tag{15}$$

subject to the initial condition

$$\hat{X}(t_2,t_2,\hat{n}) = \hat{1} \quad , \tag{16}$$

where $\hat{1}$ is the (4x4) unit matrix. Obviously, the matrix $\hat{X}$ has the property

$$\hat{X}(t_1,t',\hat{n})\,\hat{X}(t',t_2,\hat{n}) = \hat{X}(t_1,t_2,\hat{n}) \quad .$$

We express the radiation field $\hat{I}(t,\hat{n})$ for $t \in [t_t;t_b]$ in the intensity vectors $\hat{I}(t_t,\hat{m})$ and $\hat{I}(t_b,-\hat{m})$ as follows:

$$\begin{aligned}\hat{I}(t,\hat{m}) &= \hat{X}(t,t_t,\hat{m})\,\hat{I}(t_t,\hat{m}) \\ &\quad + \int d\hat{m}'\; v'\,\hat{D}(t,\hat{m},\hat{m}')\,\hat{I}(t_t,\hat{m}') \\ &\quad + \int d\hat{m}'\; v'\,\hat{U}^*(t,\hat{m},\hat{m}')\,\hat{I}(t_b,-\hat{m}') \;,\end{aligned} \tag{17}$$

$$\hat{I}(t,-\hat{m}) = \int d\hat{m}'\; v'\,\hat{U}(t,\hat{m},\hat{m}')\,\hat{I}(t_t,\hat{m}')$$

$$+ \hat{X}(t,t_b,-\hat{m})\ \hat{I}(t_b,-\hat{m})$$

$$+ \int d\hat{m}'\ v'\ \hat{D}^*(t,\hat{m},\hat{m}')\ \hat{I}(t_b,-\hat{m}') \quad . \tag{18}$$

Here, $\hat{m} = (v,\varphi)$, $-\hat{m} = (-v,\varphi)$, $v = |u|$,

$$\int d\hat{m} \quad \ldots = \int_0^1 dv \int_0^{2\pi} d\varphi \ldots \quad ,$$

$t_t = t(z_t)$, and $t_b = t(z_b)$; the matrices $\hat{U},\hat{D},\hat{U}^*$, and $\hat{D}^*$ describe the response of the scattering slab to the radiation incident on its boundaries. The reflection and transmission matrices of the slab are defined as

$$\begin{aligned}
\hat{R}(\hat{m},\hat{m}') &= \hat{U}(t_t,\hat{m},\hat{m}') \quad , \\
\hat{T}(\hat{m},\hat{m}') &= \hat{D}(t_b,\hat{m},\hat{m}') \quad , \\
\hat{R}^*(\hat{m},\hat{m}') &= \hat{U}^*(t_b,\hat{m},\hat{m}') \quad , \\
\hat{T}^*(\hat{m},\hat{m}') &= \hat{D}^*(t_t,\hat{m},\hat{m}') \quad .
\end{aligned} \tag{19}$$

Let us divide the slab $[t_t;t_b]$ into subslabs $[t_t;t]$ and $[t\ ;t_b]$. Applying Eqs. (17)-(19) recursively to the subslabs and to the combined slab $[t_t;t_b]$ (or, equivalently, using Ambartsumyan's - Chandrasekhar's principle of invariance[52,1]), we can easily express the reflection and transmission matrices of the combined slab and the internal radiation field $\hat{I}(t\ ,\hat{n})$ at the interface between the subslabs in the reflection and transmission matrices of the subslabs. We have

$$\hat{U}(t,\hat{m},\hat{m}_o) = \hat{R}_2(\hat{m},\hat{m}_o)\ \hat{X}(t,t_t,\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{R}_2(\hat{m},\hat{m}')\ \hat{D}(t,\hat{m}',\hat{m}_o) \quad , \tag{20}$$

$$\hat{D}(t,\hat{m},\hat{m}_o) = \hat{T}_1(\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{R}_1^*(\hat{m},\hat{m}')\ \hat{U}(t,\hat{m}',\hat{m}_o) \quad , \tag{21}$$

$$\hat{U}^*(t,\hat{m},\hat{m}_o) = \hat{R}_1^*(\hat{m},\hat{m}_o)\ \hat{X}(t,t_b,-\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{R}_1^*(\hat{m},\hat{m}')\ \hat{D}^*(t,\hat{m}',\hat{m}_o) \quad , \tag{22}$$

$$\hat{D}^*(t,\hat{m},\hat{m}_o) = \hat{T}_2^*(\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{R}_2(\hat{m},\hat{m}')\ \hat{U}^*(t,\hat{m}',\hat{m}_o) \quad , \tag{23}$$

$$\hat{R}(\hat{m},\hat{m}_o) = \hat{R}_1(\hat{m},\hat{m}_o) + \hat{X}(t_t,t,-\hat{m})\ \hat{U}(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{T}_1^*(\hat{m},\hat{m}')\ \hat{U}(t,\hat{m}',\hat{m}_o) \quad , \tag{24}$$

$$\hat{T}(\hat{m},\hat{m}_o) = \hat{T}_2(\hat{m},\hat{m}_o)\ \hat{X}(t,t_t,\hat{m}_o) + \hat{X}(t_b,t,\hat{m})\ \hat{D}(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{T}_2(\hat{m},\hat{m}')\ \hat{D}(t,\hat{m}',\hat{m}_o) \quad , \tag{25}$$

$$\hat{R}^*(\hat{m},\hat{m}_o) = \hat{R}_2^*(\hat{m},\hat{m}_o) + \hat{X}(t_b,t,\hat{m})\ \hat{U}^*(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{T}_2(\hat{m},\hat{m}')\ \hat{U}^*(t,\hat{m}',\hat{m}_o) \quad , \tag{26}$$

$$\hat{T}^*(\hat{m},\hat{m}_o) = \hat{T}_1^*(\hat{m},\hat{m}_o)\ \hat{X}(t,t_b,-\hat{m}_o) + \hat{X}(t_t,t,-\hat{m})\ \hat{D}^*(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{T}_1^*(\hat{m},\hat{m}')\ \hat{D}^*(t,\hat{m}',\hat{m}_o) \quad , \tag{27}$$

where by subscripts "1" and "2" we label the scattering properties of the upper and lower subslabs, respectively. The matrices $\hat{U}, \hat{D}, \hat{U}^*$, and $\hat{D}^*$, specifying the radiation field at the interface between the subslabs, can be calculated from Eqs. (20)-(23) by iterations. Thereon, the reflection and transmission matrices of the combined slab can be found from Eqs. (24)-(27).

Further, if the matrices $\hat{U}_1(t_1,\hat{m},\hat{m}')$, $\hat{D}_1(t_1,\hat{m},\hat{m}')$, $\hat{U}_1^*(t_1,\hat{m},\hat{m}')$, and $\hat{D}_1^*(t_1,\hat{m},\hat{m}')$ for $t_1 \in [t_t;t]$, and the matrices $\hat{U}_2(t_1,\hat{m},\hat{m}')$, $\hat{D}_2(t_1,\hat{m},\hat{m}')$, $\hat{U}_2^*(t_1,\hat{m},\hat{m}')$, and $\hat{D}_2^*(t_1,\hat{m},\hat{m}')$ for $t_1 \in [t;t_b]$ are known, the matrices $\hat{U}(t_1,\hat{m},\hat{m}')$, $\hat{D}(t_1,\hat{m},\hat{m}')$, $\hat{U}^*(t_1,\hat{m},\hat{m}')$, and $\hat{D}^*(t_1,\hat{m},\hat{m}')$ for the combined slab can also be easily calculated. Using Eqs. (17) and (18), we find

$$\begin{aligned}\hat{U}(t_1,\hat{m},\hat{m}_o) &= \hat{U}_1(t_1,\hat{m},\hat{m}_o) \\ &\quad + \hat{X}(t_1,t,-\hat{m})\,\hat{U}(t,\hat{m},\hat{m}_o) \\ &\quad + \int d\hat{m}'\, v'\, \hat{D}_1^*(t_1,\hat{m},\hat{m}')\,\hat{U}(t,\hat{m}',\hat{m}_o) \quad , \end{aligned} \tag{28}$$

$$\begin{aligned}\hat{D}(t_1,\hat{m},\hat{m}_o) &= \hat{D}_1(t_1,\hat{m},\hat{m}_o) \\ &\quad + \int d\hat{m}'\, v'\, \hat{U}_1^*(t_1,\hat{m},\hat{m}')\,\hat{U}(t,\hat{m}',\hat{m}_o) \quad , \end{aligned} \tag{29}$$

$$\begin{aligned}\hat{U}^*(t_1,\hat{m},\hat{m}_o) &= \hat{U}_1^*(t_1,\hat{m},\hat{m}_o)\,\hat{X}(t,t_b,-\hat{m}_o) \\ &\quad + \int d\hat{m}'\, v'\, \hat{U}_1^*(t_1,\hat{m},\hat{m}')\,\hat{D}^*(t,\hat{m}',\hat{m}_o) \quad , \end{aligned} \tag{30}$$

$$\begin{aligned}\hat{D}^*(t_1,\hat{m},\hat{m}_o) &= \hat{D}_1^*(t_1,\hat{m},\hat{m}_o)\,\hat{X}(t,t_b,-\hat{m}_o) \\ &\quad + \hat{X}(t_1,t,-\hat{m})\,\hat{D}^*(t,\hat{m},\hat{m}_o) \\ &\quad + \int d\hat{m}'\, v'\, \hat{D}_1^*(t_1,\hat{m},\hat{m}')\,\hat{D}^*(t,\hat{m}',\hat{m}_o) \end{aligned} \tag{31}$$

for $t_1 \in [t_t;t]$, and

$$\hat{U}(t_1,\hat{m},\hat{m}_o) = \hat{U}_2(t_1,\hat{m},\hat{m}_o)\ \hat{X}(t,t_t,\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{U}_2(t_1,\hat{m},\hat{m}')\ \hat{D}(t,\hat{m}',\hat{m}_o) \quad , \tag{32}$$

$$\hat{D}(t_1,\hat{m},\hat{m}_o) = \hat{D}_2(t_1,\hat{m},\hat{m}_o)\ \hat{X}(t,t_t,\hat{m}_o) + \hat{X}(t_1,t,\hat{m})\ \hat{D}(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{D}_2(t_1,\hat{m},\hat{m}')\ \hat{D}(t,\hat{m}',\hat{m}_o) \quad , \tag{33}$$

$$\hat{U}^*(t_1,\hat{m},\hat{m}_o) = \hat{U}_2^*(t_1,\hat{m},\hat{m}_o) + \hat{X}(t_1,t,\hat{m})\ \hat{U}^*(t,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{D}_2(t_1,\hat{m},\hat{m}')\ \hat{U}^*(t,\hat{m}',\hat{m}_o) \quad , \tag{34}$$

$$\hat{D}^*(t_1,\hat{m},\hat{m}_o) = \hat{D}_2^*(t_1,\hat{m},\hat{m}_o) + \int d\hat{m}'\ v'\ \hat{U}_2(t_1,\hat{m},\hat{m}')\ \hat{U}^*(t,\hat{m}',\hat{m}_o) \tag{35}$$

for $t_1 \in [t;t_b]$.

If the scattering medium is isotropic, we have

$$\hat{K}(t,\hat{n}) = -C_{ext}(\tau)\ \hat{1} \ , \tag{36}$$

$$\hat{X}(t_1,t_2,\hat{n}) = \exp[-(\tau_1-\tau_2)/u]\ \hat{1} \quad , \tag{37}$$

where C_{ext} is the extinction cross-section, and

$$\tau(t) = \int_0^t dt'\ C_{ext}(t') \tag{38}$$

is the optical depth. Inserting Eqs. (36)-(37) into Eqs. (17)-(18) and (20)-(35), we obtain the well known

equations derived, e.g., in Refs. 1,3,5,7,9,16, and 19.

It should be noted that convergence of the iterative solution of Eqs. (20)-(23) was proved only for homogeneous isotropic subslabs.[14] However, from physical considerations we can expect the convergence even for inhomogeneous anisotropic subslabs.

4. INVARIANT IMBEDDING EQUATIONS

Assuming $\Delta t = t - t_t \ll 1$, using the equation of transfer (14) and Eqs. (17)-(23), and neglecting all terms proportional to $(\Delta t)^n$ for $n > 1$, we have

$$\hat{R}_1(\hat{m},\hat{m}') = \hat{Z}(t_t,-\hat{m},\hat{m}')\,\Delta t/(vv') \quad ,$$

$$\hat{R}_1^*(\hat{m},\hat{m}') = \hat{Z}(t_t,\hat{m},-\hat{m}')\,\Delta t/(vv') \quad ,$$

$$\hat{T}_1(\hat{m},\hat{m}') = \hat{Z}(t_t,\hat{m},\hat{m}')\,\Delta t/(vv') \quad ,$$

$$\hat{T}_1^*(\hat{m},\hat{m}') = \hat{Z}(t_t,-\hat{m},-\hat{m}')\,\Delta t/(vv') \quad ,$$

$$\hat{X}(t,t_t,\hat{m}_o) = \hat{1} + \Delta t\,\hat{K}(t_t,\hat{m}_o)/v_o \quad , \tag{39}$$

$$\hat{X}(t_t,t,-\hat{m}) = \hat{1} + \Delta t\,\hat{K}(t_t,-\hat{m})/v \quad , \tag{40}$$

$$\begin{aligned}\hat{U}(t,\hat{m},\hat{m}_o) = {} & \hat{R}_2(\hat{m},\hat{m}_o) \\ & + \frac{\Delta t}{v_o}\,\hat{R}(\hat{m},\hat{m}_o)\,\hat{K}(t_t,\hat{m}_o) \\ & + \frac{\Delta t}{v_o}\int d\hat{m}'\,\hat{R}(\hat{m},\hat{m}')\,\hat{Z}(t_t,\hat{m}',\hat{m}_o) \\ & + \Delta t\int d\hat{m}'\int d\hat{m}''\,\hat{R}(\hat{m},\hat{m}')\,\hat{Z}(t_t,\hat{m}',-\hat{m}'') \\ & \cdot \hat{R}(\hat{m}'',\hat{m}_o) \quad ,\end{aligned} \tag{41}$$

$$\hat{D}(t,\hat{m},\hat{m}_o) = \frac{\Delta t}{vv_o}\ \hat{Z}(t_t,\hat{m},\hat{m}_o) + \frac{\Delta t}{v}\int d\hat{m}'\ \hat{Z}(t_t,\hat{m},-\hat{m}')\ \hat{R}(\hat{m}',\hat{m}_o)\ , \tag{42}$$

$$\hat{U}^*(t,\hat{m},\hat{m}_o) = \frac{\Delta t}{vv_o}\ \hat{Z}(t_t,\hat{m},-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o) + \frac{\Delta t}{v}\int d\hat{m}'\ \hat{Z}(t_t,\hat{m},-\hat{m}')\ \hat{T}^*(\hat{m}',\hat{m}_o)\ , \tag{43}$$

$$\begin{aligned}\hat{D}^*(t,\hat{m},\hat{m}_o) = {} & \hat{T}_2^*(\hat{m},\hat{m}_o) \\ & + \frac{\Delta t}{v_o}\int d\hat{m}'\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o) \\ & + \Delta t\int d\hat{m}'\int d\hat{m}''\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}'') \\ & \cdot\ \hat{T}^*(\hat{m}'',\hat{m}_o)\ .\end{aligned} \tag{44}$$

Inserting Eqs. (39)-(44) into Eqs. (24)-(27), we derive

$$\begin{aligned}\frac{d\ \hat{R}(\hat{m},\hat{m}_o)}{dt_t} = {} & -\ \hat{R}(\hat{m},\hat{m}_o)\ \hat{K}(t_t,\hat{m}_o)/v_o \\ & -\ \hat{K}(t_t,-\hat{m})\ \hat{R}(\hat{m},\hat{m}_o)/v \\ & -\ \frac{1}{vv_o}\ \hat{Z}(t_t,-\hat{m},\hat{m}_o) \\ & -\ \frac{1}{v_o}\int d\hat{m}'\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',\hat{m}_o) \\ & -\ \frac{1}{v}\int d\hat{m}'\ \hat{Z}(t_t,-\hat{m},-\hat{m}')\ \hat{R}(\hat{m}',\hat{m}_o) \\ & -\ \int d\hat{m}'\int d\hat{m}''\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}'') \\ & \cdot\ \hat{R}(\hat{m}'',\hat{m}_o)\ ,\end{aligned} \tag{45a}$$

$$\frac{d\ \hat{T}(\hat{m},\hat{m}_o)}{dt_t} = -\ \hat{T}(\hat{m},\hat{m}_o)\ \hat{K}(t_t,\hat{m}_o)/v_o$$

$$- \frac{1}{vv_o} \hat{X}(t_b,t_t,\hat{m})\ \hat{Z}(t_t,\hat{m},\hat{m}_o)$$

$$- \frac{1}{v_o} \int d\hat{m}'\ \hat{T}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',\hat{m}_o)$$

$$- \frac{1}{v} \hat{X}(t_b,t_t,\hat{m}) \int d\hat{m}'\ \hat{Z}(t_t,\hat{m},-\hat{m}')\ \hat{R}(\hat{m}',\hat{m}_o)$$

$$- \int d\hat{m}' \int d\hat{m}''\ \hat{T}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}'')$$

$$\cdot\ \hat{R}(\hat{m}'',\hat{m}_o) \quad , \tag{46a}$$

$$\frac{d\ \hat{R}^*(\hat{m},\hat{m}_o)}{dt_t} = - \frac{1}{vv_o} \hat{X}(t_b,t_t,\hat{m})\ \hat{Z}(t_t,\hat{m},-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o)$$

$$- \frac{1}{v} \hat{X}(t_b,t_t,\hat{m}) \int d\hat{m}'\ \hat{Z}(t_t,\hat{m},-\hat{m}')\ \hat{T}^*(\hat{m}',\hat{m}_o)$$

$$- \frac{1}{v_o} \int d\hat{m}'\ \hat{T}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o)$$

$$- \int d\hat{m}' \int d\hat{m}''\ \hat{T}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}'')\ \hat{T}^*(\hat{m}'',\hat{m}_o), \tag{47a}$$

$$\frac{d\ \hat{T}^*(\hat{m},\hat{m}_o)}{dt_t} = - \hat{K}(t_t,-\hat{m})\ \hat{T}^*(\hat{m},\hat{m}_o)/v$$

$$- \frac{1}{vv_o} \hat{Z}(t_t,-\hat{m},-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o)$$

$$- \frac{1}{v} \int d\hat{m}'\ \hat{Z}(t_t,-\hat{m},-\hat{m}')\ \hat{T}^*(\hat{m}',\hat{m}_o)$$

$$- \frac{1}{v_o} \int d\hat{m}'\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}_o)\ \hat{X}(t_t,t_b,-\hat{m}_o)$$

$$- \int d\hat{m}' \int d\hat{m}''\ \hat{R}(\hat{m},\hat{m}')\ \hat{Z}(t_t,\hat{m}',-\hat{m}'')$$

$$\cdot\ \hat{T}^*(\hat{m}'',\hat{m}_o) \quad . \tag{48a}$$

Similarly, by setting $\Delta t = t_b - t \ll 1$, we can derive a system of 4 equations, which have in the left-hand

side the derivatives $d\hat{R}/dt_b$, $d\hat{T}/dt_b$, $d\hat{R}^*/dt_b$, and $d\hat{T}^*/dt_b$, respectively. For brevity, these equations are not given here; in what follows, they will be referred to as Eqs. (45b)-(48b).

Eqs. (45a)-(48a) and (45b)-(48b) form the complete set of invariant imbedding equations for the reflection and transmission matrices. They should be supplemented with the initial conditions

$$\begin{aligned} \hat{R}(\hat{m},\hat{m}_o)\Big|_{t_t=t_b} &= \hat{O} \ , \\ \hat{T}(\hat{m},\hat{m}_o)\Big|_{t_t=t_b} &= \hat{O} \ , \\ \hat{R}^*(\hat{m},\hat{m}_o)\Big|_{t_t=t_b} &= \hat{O} \ , \\ \hat{T}^*(\hat{m},\hat{m}_o)\Big|_{t_t=t_b} &= \hat{O} \ , \end{aligned} \tag{49}$$

where $\hat{O}$ is the (4x4) zero matrix.

For an isotropic scattering medium, we may use Eqs. (36)-(38) and the definitions

$$\hat{Z}_i(\tau,u,u',\varphi-\varphi') = \frac{4\pi}{C_{sca}}\,\hat{Z}(t,\hat{n},\hat{n}') \ , \quad w = \frac{C_{sca}}{C_{ext}} \ , \tag{50}$$

where C_{sca} is the scattering cross-section, and w is the single-scattering albedo, to reduce the invariant imbedding equations to the form derived earlier in Refs. 1 and 5.

Assuming uniqueness of solution of the invariant imbedding equations (45a)-(48a) (or (45b)-(48b)), subject to the initial conditions Eqs. (49), we easily find that the reflection and transmission matrices obey the reciprocity relations

$$\hat{R}(\hat{m},\hat{m}_o) = \hat{P}\,\hat{R}^T(\hat{m}_o,\hat{m})\,\hat{P} \ , \tag{51}$$

$$\hat{R}^*(\hat{m},\hat{m}_o) = \hat{P}\ \hat{R}^{*T}(\hat{m}_o,\hat{m})\ \hat{P} \quad , \tag{52}$$

$$\hat{T}(\hat{m},\hat{m}_o) = \hat{P}\ \hat{T}^{*T}(\hat{m}_o,\hat{m})\ \hat{P} \quad . \tag{53}$$

For homogeneous slabs, a system of 4 nonlinear integral equations for the reflection and transmission matrices can be obtained by taking the sum of conjugate equations (45a)-(45b), (46a)-(46b), (47a)-(47b), and (48a)-(48b), respectively (this system of equations is not given here for brevity). Owing to the symmetry relations[5,19]

$$\hat{R}_i(v,v',\varphi-\varphi') = \hat{R}_i^*(v,v',\varphi'-\varphi) \quad , \tag{54}$$

$$\hat{T}_i(v,v',\varphi-\varphi') = \hat{T}_i^*(v,v',\varphi'-\varphi) \quad , \tag{55}$$

the number of independent equations for isotropic homogeneous slabs is only 2. These equations may be found in Refs. 5,12, and 18. By setting $t_b = \infty$ and $d\hat{R}/dt_t = \hat{0}$, we obtain from Eq. (45a) the nonlinear integral equation for the reflection matrix of a semi-infinite, homogeneous anisotropic slab:

$$\begin{aligned} -v\ \hat{R}_\infty(\hat{m},\hat{m}_o)\ \hat{K}(\hat{m}_o) - v_o\ \hat{K}(-\hat{m})\ \hat{R}_\infty(\hat{m},\hat{m}_o) \\ = \hat{Z}(-\hat{m},\hat{m}_o) \\ + v \int d\hat{m}'\ \hat{R}_\infty(\hat{m},\hat{m}')\ \hat{Z}(\hat{m}',\hat{m}_o) \\ + v_o \int d\hat{m}'\ \hat{Z}(-\hat{m},-\hat{m}')\ \hat{R}_\infty(\hat{m}',\hat{m}_o) \\ + vv_o \int d\hat{m}' \int d\hat{m}''\ \hat{R}_\infty(\hat{m},\hat{m}')\ \hat{Z}(\hat{m}',-\hat{m}'') \\ \cdot\ \hat{R}_\infty(\hat{m}'',\hat{m}_o) \quad . \end{aligned} \tag{56}$$

For isotropic media, analogous equations were derived previously by Ambartsumyan[53] for the scalar case and Domke[6] for the vector case.

One would expect that the nonlinear integral equations for the reflection and transmission matrices of a homogeneous anisotropic slab permit not only the physically relevant solutions. The nonuniqueness problem in solving the nonlinear integral equations for isotropic media was extensively studied by De Rooij and Domke.[12]

5. DISCUSSION

As stated above, the adding equations (20)-(35) and the invariant imbedding equations (45)-(48) are a generalized version of equations, which have been derived earlier for isotropic plane-parallel media. It is easily seen that all differences of Eqs. (20)-(35) and (45)-(48) from the corresponding equations for isotropic media are due to the following causes.

(i) In the transfer equation (14), the extinction matrix occurs in place of the scalar extinction cross-section. As a result, the single-scattering albedo $w = C_{sca}/C_{ext}$ and the optical depth (thickness) τ can not be introduced, and the matrix $\hat{X}(t_1,t_2,\hat{n})$ appears instead of a quantity $\exp[-(\tau_1-\tau_2)/u]$.

(ii) Generally, the phase matrix $\hat{Z}(\hat{n},\hat{n}')$ depends on each of the azimuth angles φ and φ', whereas the phase matrix of an isotropic medium depends only on their difference. Therefore, the Fourier analysis can not be effectively used to handle the azimuth dependence of the phase matrix, reflection and transmission matrices, and so on.

In addition, it should be noted that, generally, only one of three symmetry relations for the matrix $\hat{Z}_i$

(see e.g. Ref. 11) is satisfied by the matrix $\hat{Z}$, namely, the reciprocity relation Eq. (8). As a result, the symmetry relations[5,18]

$$\hat{R}_i(v,v',\varphi-\varphi') = \hat{D}\,\hat{R}_i(v,v',\varphi'-\varphi)\,\hat{D}, \;,$$

$$\hat{T}_i(v,v',\varphi-\varphi') = \hat{D}\,\hat{T}_i(v,v',\varphi'-\varphi)\,\hat{D} \;,$$

where $\hat{D} = \mathrm{diag}(1,1,-1,-1)$, are also lost as well as the symmetry relations Eqs. (54)-(55) for the reflection and transmission matrices of a homogeneous isotropic slab.

The adding equations and the invariant imbedding equations can be used to calculate the intensity vector of light scattered in arbitrary vertically-inhomogeneous anisotropic slab. The general computational scheme may be as follows.

Consider a slab $[t_t;t_b]$, which is illuminated by radiation specified by intensity vectors $\hat{I}(t_t,\hat{m})$ and $\hat{I}(t_b,-\hat{m})$. The problem is to calculate the intensity vector $\hat{I}(t,\hat{n})$ for $t \in [t_t;t_b]$.

It follows from Eqs. (17) and (18) that the problem is reduced to the calculation of the matrices $\hat{U}(t),\hat{D}(t),\hat{U}^*(t),\hat{D}^*(t),\hat{X}(t,t_t)$, and $\hat{X}(t,t_b)$. For calculating the matrices $\hat{X}(t,t_t)$ and $\hat{X}(t,t_b)$, Eq. (15) is used subject to the initial condition Eq. (16). The matrices $\hat{U}(t),\hat{D}(t),\hat{U}^*(t)$, and $\hat{D}^*(t)$ are calculated from Eqs. (20)-(23) by iterations. Finally, the reflection and transmission matrices for the subslabs $[t_t;t]$ and $[t;t_b]$, occurring in Eqs. (20)-(23), are calculated by a numerical solution of the invariant imbedding equations (45a)-(48a) or (45b)-(48b).

The problem becomes much simpler if the slab can be divided into a number of homogeneous subslabs. In

that case, we can use computational schemes, which are entirely based on the adding equations. These computational schemes can be developed as a direct extension of those proposed for isotropic slabs (see e.g. Refs. 3,5,7,9,13,16,19,21,54-57 and references therein).

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