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MULTIPLE LIGHT SCATTERING BY POLYDISPERSIONS OF RANDOMLY DISTRIBUTED, PERFECTLY-ALIGNED, INFINITE MIE CYLINDERS ILLUMINATED PERPENDICULARLY TO THEIR AXES

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Abstract—The radiative transfer theory has been applied to calculate multiple light scattering by ensembles of randomly distributed, perfectly aligned, polydisperse, infinite Mie cylinders illuminated perpendicularly to their axes. An efficient method for computing the single-scattering Stokes matrix for polydispersions of Mie cylinders is described. The albedo problem for a homogeneous half-space of scatterers is considered. Illustrative numerical results for the backscattering coefficients are computed for polydispersions of cylinders with different refractive indices. An application to the problem of weak localization of photons is given. Specifically, enhanced backscattering of light by two-dimensional, discrete, disordered media is considered and backscattering enhancement factors in exactly the backscattering direction are computed for a number of scattering models.

1. INTRODUCTION

In recent years, there has been growing interest in multiple scattering of electromagnetic waves by two-dimensional, discrete media composed of infinitely long, perfectly aligned cylinders illuminated perpendicularly to their axes (see, e.g., Refs. 1–9 and references therein). In this case, the general vector problem is easily decomposed into a number of scalar subproblems, which makes its solution rather simple and transparent.

If the cylinders are randomly distributed and independently scattering and the ladder approximation of the Bethe–Salpeter equation is used, then the multiple scattering of light can be described by an appropriate radiative transfer equation. This equation was first formulated in Refs. 3 and 4. Subsequently, in Refs. 3–5, classical methods of radiative transfer theory[10–13] were used to study analytical solutions of this equation extensively, and some numerical results were reported for the simplest case of Rayleigh scattering (radii of the cylinders are much smaller than the wavelength).

The purpose of the present paper is to consider a more general type of scattering, namely, scattering by polydispersions of circular cylinders with radii comparable to the wavelength (hereafter Mie cylinders) and to produce some illustrative numerical results. In Sec. 2, basic definitions and equations are briefly introduced. In Sec. 3, we describe an efficient analytical method for computing the elements of the size-averaged single-scattering Stokes matrix, which must be determined before the radiative transfer equation is solved numerically. In this method, the elements of the scattering matrix are expanded in Fourier series and the corresponding expansion coefficients are expressed directly in the coefficients that appear in the Mie series for cylinders.[14,15] We average over sizes the expansion coefficients rather than the elements of the scattering matrix, which is advantageous in numerical calculations if the elements of the scattering matrix have to be computed for a great number of scattering angles. In other words, our method is an application of the idea by Domke[16] and Bugaenko,[17] which was first used to solve the Mie problem for polydispersions of spheres. In Sec. 4, we consider the albedo problem for homogeneous semi-infinite media. To compute the Stokes parameters of the reflected light, we solve numerically Ambartsumian's[18] nonlinear integral equation for the reflection matrix. Illustrative numerical results for the backscattering coefficients are computed for polydispersions of cylinders with

different refractive indices. These numerical data are compared with analogous computations for very thin (Rayleigh) cylinders. Finally, in Sec. 5, an application to the problem of weak localization of photons is given. Specifically, enhanced backscattering of light by 2-D discrete disordered media is considered and backscattering enhancement factors in exactly the backscattering direction are computed for a number of scattering models.

2. BASIC DEFINITIONS AND EQUATIONS

We consider multiple light scattering in a plane-parallel medium composed of perfectly aligned, randomly distributed, infinite, circular cylinders. The medium is illuminated by a parallel beam of light, which is incident perpendicularly to the cylinder axes. This multiple-scattering problem is essentially 2-D, because the unit vector $\hat{\mathbf{n}}$ that specifies the direction of the multiply-scattered light always lies in the plane perpendicular to the cylinder axes.

To describe the scattering of light, we use the cylindrical polar coordinate system that is shown in Fig. 1. The X-axis is directed towards the inward normal to the upper boundary of the medium and the plane XOY is perpendicular to the cylinder axes. Thus, the Z-axis is parallel to the cylinder axes and is directed out of the paper. The plane YOZ coincides with the upper boundary of the medium. The direction of light propagation $\hat{\mathbf{n}}$ is specified by the azimuth angle φ. We use the Stokes parameters that are defined by

$$I_1 = \langle E_z E_z^* \rangle, \tag{1}$$

$$I_2 = \langle E_\varphi E_\varphi^* \rangle, \tag{2}$$

$$U = 2\mathrm{Re}\langle E_z E_\varphi^* \rangle, \tag{3}$$

$$V = -2\mathrm{Im}\langle E_z E_\varphi^* \rangle, \tag{4}$$

where the asterisk denotes the conjugate complex value, the angle brackets denote the ensemble average, and E_z and E_φ are the components of the electric field

$$\mathbf{E} = E_z \hat{\mathbf{e}}_z + E_\varphi \hat{\mathbf{e}}_\varphi. \tag{5}$$

Here, $\hat{\mathbf{e}}_z$ and $\hat{\mathbf{e}}_\varphi$ are the corresponding unit vectors. The Stokes vector is defined as a (4×1) column which has the Stokes parameters as its components as follows:

$$\mathbf{I} = \{I_1, I_2, U, V\} = [I_1, I_2, U, V]^{\mathrm{T}} = \begin{bmatrix} I_1 \\ I_2 \\ U \\ V \end{bmatrix}, \tag{6}$$

where T denotes matrix transpose. An external parallel beam of light is incident perpendicularly to the cylinder axes and is specified by the angle of incidence φ_0 and the Stokes vector $\mathbf{I}^0$.

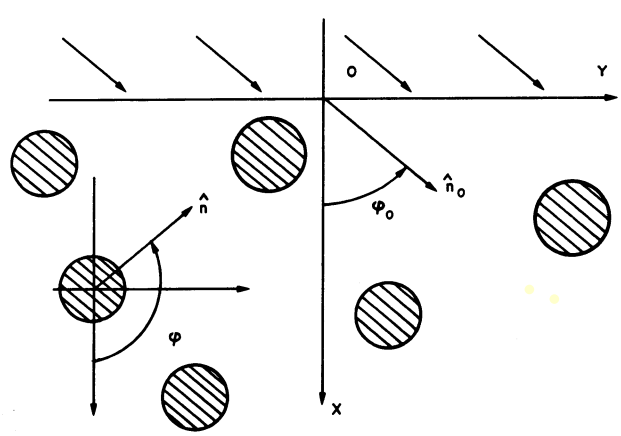

Fig. 1. Geometry of light scattering.

As is well known (see, e.g., Ref. 19 and references therein), the Bethe–Salpeter equation under the ladder approximation of uncorrelated, independent scatterers results in the common vector radiative transfer equation.[10,20] For the 2-D medium considered, this equation has the form[3,4]

$$\cos\varphi \frac{\mathrm{d}\mathbf{I}(x;\varphi)}{\mathrm{d}x} = -\rho(x)\mathbf{K}(x)\mathbf{I}(x;\varphi) + \rho(x)\int_0^{2\pi} \mathrm{d}\varphi'\, \mathbf{Z}(x;\varphi-\varphi')\mathbf{I}(x;\varphi'), \tag{7}$$

where $\mathbf{I}(x;\varphi)$ is the Stokes vector of the multiply-scattered light, $\rho(x)$ the concentration of the cylinders in the plane perpendicular to the cylinder axes, $\mathbf{K}(x)$ the extinction matrix, and $\mathbf{Z}(x;\varphi)$ the scattering (or phase) matrix.

The elements of the matrices $\mathbf{K}$ and $\mathbf{Z}$ can be expressed in terms of the elements of the amplitude scattering matrix. For a normally illuminated cylinder, the amplitude scattering matrix is diagonal[14,15] and

$$\begin{bmatrix} E_z^{\mathrm{s}} \\ E_\varphi^{\mathrm{s}} \end{bmatrix} = \mathrm{e}^{i3\pi/4}\left(\frac{2}{\pi k R}\right)^{1/2} \mathrm{e}^{ikR} \begin{bmatrix} T_1(\theta) & 0 \\ 0 & T_2(\theta) \end{bmatrix} \begin{bmatrix} E_z^{\mathrm{i}} \\ E_\varphi^{\mathrm{i}} \end{bmatrix}, \tag{8}$$

where R is the distance from the cylinder, k a free-space wavenumber, θ the scattering angle, the superscript s denotes the scattered field components, and the superscript i denotes the incident field components. Throughout this paper, a time dependence of the form $\exp(-i\omega t)$ is assumed and suppressed. As a result, we have[3,4]

$$\mathbf{K} = \frac{2}{k}\begin{bmatrix} 2\mathrm{Re}\langle T_1(0)\rangle & 0 & 0 & 0 \\ 0 & 2\mathrm{Re}\langle T_2(0)\rangle & 0 & 0 \\ 0 & 0 & \mathrm{Re}\langle T_1(0)+T_2(0)\rangle & \mathrm{Im}\langle T_1(0)-T_2(0)\rangle \\ 0 & 0 & -\mathrm{Im}\langle T_1(0)-T_2(0)\rangle & \mathrm{Re}\langle T_1(0)+T_2(0)\rangle \end{bmatrix}, \tag{9}$$

$$\mathbf{Z}(\theta) = \frac{2}{\pi k}\begin{bmatrix} \langle |T_1(\theta)|^2\rangle & 0 & 0 & 0 \\ 0 & \langle |T_2(\theta)|^2\rangle & 0 & 0 \\ 0 & 0 & \mathrm{Re}\langle T_1(\theta)T_2^*(\theta)\rangle & \mathrm{Im}\langle T_1(\theta)T_2^*(\theta)\rangle \\ 0 & 0 & -\mathrm{Im}\langle T_1(\theta)T_2^*(\theta)\rangle & \mathrm{Re}\langle T_1(\theta)T_2^*(\theta)\rangle \end{bmatrix}, \tag{10}$$

where the angle brackets denote the average over the cylinder radii.

3. SINGLE SCATTERING BY SIZE DISTRIBUTIONS OF MIE CYLINDERS

Before solving the radiative transfer equation (7) numerically, the elements of the scattering matrix $\mathbf{Z}(\theta)$ must be calculated for a great number of scattering angles. In this section, we describe an efficient analytical method for computing the elements of this matrix for a size distribution of circular cylinders. The method is an application of an idea by Domke[16] and Bugaenko,[17] which was used earlier to solve the Mie problem for polydisperse spherical particles.[21,22]

The elements of the amplitude scattering matrix are given by the Mie series[14,15]

$$T_1(\theta) = \sum_{n=-\infty}^{\infty} b_n \exp(in\theta), \quad b_{-n} = b_n, \tag{11}$$

$$T_2(\theta) = \sum_{n=-\infty}^{\infty} a_n \exp(in\theta), \quad a_{-n} = a_n, \tag{12}$$

where the Mie coefficients b_n and a_n depend on the refractive index N and size parameter $x = kr$ (r is the cylinder radius). We use the following similar trigonometric expansions for the elements of the size-averaged scattering matrix:

$$Z_{11}(\theta) = \frac{1}{2\pi} C_{\text{sca},1} \sum_{l=-\infty}^{\infty} y_l \exp(il\theta), \quad y_{-l} = y_l, \tag{13}$$

$$Z_{22}(\theta) = \frac{1}{2\pi} C_{\text{sca},2} \sum_{l=-\infty}^{\infty} x_l \exp(il\theta), \quad x_{-l} = x_l, \tag{14}$$

$$Z_{33}(\theta) + iZ_{34}(\theta) = \frac{1}{2\pi} C \sum_{l=-\infty}^{\infty} z_l \exp(il\theta), \quad z_{-l} = z_l, \tag{15}$$

where $C_{\text{sca},1}$ and $C_{\text{sca},2}$ are the corresponding scattering cross sections given by[14]

$$C_{\text{sca},1} = \frac{2}{\pi k} \int_0^{2\pi} \mathrm{d}\theta \, |T_1(\theta)|^2 = \frac{4}{k} \sum_{n=-\infty}^{\infty} \langle |b_n|^2 \rangle, \tag{16}$$

$$C_{\text{sca},2} = \frac{2}{\pi k} \int_0^{2\pi} \mathrm{d}\theta \, |T_2(\theta)|^2 = \frac{4}{k} \sum_{n=-\infty}^{\infty} \langle |a_n|^2 \rangle, \tag{17}$$

and the factor C is given by

$$C = \frac{2}{\pi k} \int_0^{2\pi} \mathrm{d}\theta \, T_1(\theta) T_2^*(\theta) = \frac{4}{k} \sum_{n=-\infty}^{\infty} \langle b_n a_n^* \rangle. \tag{18}$$

It should be noted that $y_0 \equiv 1$, $x_0 \equiv 1$, and $z_0 \equiv 1$. By substituting the Mie series (11) and (12) into Eq. (10) and using Eqs. (13)–(15), we easily express the expansion coefficients y_l, x_l, and z_l directly in terms of the Mie coefficients b_n and a_n, viz.

$$y_l = \frac{4}{kC_{\text{sca},1}} \sum_{n=-\infty}^{\infty} \langle b_n b_{n-l}^* \rangle, \tag{19}$$

$$x_l = \frac{4}{kC_{\text{sca},2}} \sum_{n=-\infty}^{\infty} \langle a_n a_{n-l}^* \rangle, \tag{20}$$

$$z_l = \frac{4}{kC} \sum_{n=-\infty}^{\infty} \langle b_n a_{n-l}^* \rangle. \tag{21}$$

The expansions (19)–(21) are very useful in numerical computations for size distributions of cylinders, because one may average over sizes the expansion coefficients y_l, x_l, and z_l rather than the elements of the scattering matrix. Next, by using Eqs. (13)–(15), the elements of the scattering matrix can be computed for a great number of scattering angles with a minimum expense of computer time.

In Tables 1 and 2, we give an example of using Eqs. (13)–(21). The computations are reported for a standard gamma distribution[23] of cylinder radii

$$n(r) = \text{constant}\, r^{(1-3v_{\text{eff}})/v_{\text{eff}}} \exp[-r/(r_{\text{eff}} v_{\text{eff}})], \tag{22}$$

with $r_{\text{eff}} = 0.3\,\mu\text{m}$ and $v_{\text{eff}} = 0.05$. The refractive index is $N = 1.32$ and the free-space wavelength is $\lambda = 0.5\,\mu\text{m}$. To compute the Mie coefficients, we used the subroutine BHCYL of Bohren and Huffman.[15] In Table 1, the size-averaged expansion coefficients y_l, x_l, and z_l are given. In Table 2, these expansion coefficients are used to compute the quantities

$$\chi_1(\theta) = 2\pi Z_{11}(\theta)/C_{\text{sca},1}, \tag{23}$$

$$\chi_2(\theta) = 2\pi Z_{22}(\theta)/C_{\text{sca},2}, \tag{24}$$

$$\chi(\theta) = 2\pi [Z_{33}(\theta) + iZ_{34}(\theta)]/C. \tag{25}$$

Table 1. Expansion coefficients for the gamma distribution of Mie cylinders with $r_{\text{eff}} = 0.3\ \mu\text{m}$, $v_{\text{eff}} = 0.05$, $N = 1.32$, and $\lambda = 0.5\ \mu\text{m}$.

l	y_l	x_l	Re z_l	Im z_l
0	1.00000	1.00000	1.00000	0.00000
1	0.87838	0.91499	0.94422	0.83397
2	0.74158	0.76233	0.77212	0.63825
3	0.57714	0.58539	0.60449	0.68408
4	0.42615	0.42604	0.43833	0.53947
5	0.28685	0.28903	0.29841	0.52129
6	0.18093	0.18355	0.18607	0.37192
7	0.09743	0.10708	0.10547	0.27017
8	0.05195	0.05760	0.05512	0.15177
9	0.02188	0.02808	0.02547	0.08206
10	0.01014	0.01271	0.01128	0.03691
11	0.00341	0.00518	0.00430	0.01557
12	0.00142	0.00201	0.00166	0.00593
13	0.00039	0.00070	0.00054	0.00204
14	0.00015	0.00024	0.00018	0.00069
15	0.00004	0.00007	0.00005	0.00020
16	0.00001	0.00002	0.00002	0.00006
17	0.00000	0.00001	0.00000	0.00002
18	0.00000	0.00000	0.00000	0.00000

Table 2. Normalized elements of the Stokes scattering matrix for the gamma distribution of Mie cylinders with $r_{\text{eff}} = 0.3\ \mu\text{m}$, $v_{\text{eff}} = 0.05$, $N = 1.32$, and $\lambda = 0.5\ \mu\text{m}$.

θ	χ_1	χ_2	Re χ	Im χ
0°	7.55571	7.75004	7.89545	9.30877
15°	5.21078	5.30087	5.43078	4.97432
30°	1.77713	1.83742	1.86557	0.35665
45°	0.43256	0.49947	0.46713	0.16114
60°	0.20307	0.20212	0.19135	0.49893
75°	0.13816	0.10047	0.10258	0.39198
90°	0.09346	0.05367	0.05094	0.29892
105°	0.07128	0.03240	0.02220	0.27886
120°	0.06637	0.02228	0.00548	0.27818
135°	0.06503	0.01883	-0.00656	0.26459
150°	0.05235	0.01962	-0.02296	0.14844
165°	0.06516	0.02387	-0.03604	-0.14202
180°	0.09360	0.02792	-0.03638	-0.32875

All these quantities are normalized to unity as follows:

$$(1/2\pi)\int_0^{2\pi}\begin{Bmatrix}\chi_1(\theta)\\ \chi_2(\theta)\\ \chi(\theta)\end{Bmatrix}\mathrm{d}\theta = 1. \tag{26}$$

In computing the averaged quantities by integrating over the size distribution, we split the integration interval [0, 2] μm into 10 subintervals, in each of which a 24-point Gauss quadrature formula was used. Both the integration interval and the number of quadrature division points were chosen to ensure an accuracy of 1×10^{-5} of the computed expansion coefficients. The expansion coefficients were computed in 23 sec, and then 0.01 sec were required to compute the scattering matrix for one scattering angle. In these computations, a computer ES 1061 was used. For comparison, the common numerical method, which implies size averaging of the elements of the scattering matrix themselves, required 4 sec for the computation of the Mie coefficients and then 0.7 sec to compute the elements of the scattering matrix for one scattering angle. Thus, our analytical method is more efficient if the number of scattering angles, for which the scattering matrix is computed, is greater than roughly 30. For example, to solve Eq. (34) numerically, we had to compute the elements of the scattering matrix for 690 scattering angles, and our method was roughly 16 times faster than the commonly used numerical method.

4. ELECTROMAGNETIC BACKSCATTERING BY A HOMOGENEOUS HALF-SPACE OF INFINITE CYLINDERS

Let a plane-parallel medium be illuminated by a parallel beam of light with $\mathbf{I}^0 = \pi\,\{I_1^0, I_2^0, 0, 0\}$. It follows from Eqs. (7), (9), and (10) that the Stokes parameters U and V of the multiply-scattered light are identically equal to zero, and the vector radiative transfer equation (7) is reduced to the following system of two independent scalar equations:[3,4]

$$\cos\varphi\,\frac{\mathrm{d}I_j(\tau_j;\varphi)}{\mathrm{d}\tau_j} = -I_j(\tau_j;\varphi) + \frac{w_j(\tau_j)}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi'\,\chi_j(\tau_j;\varphi-\varphi')I_j(\tau_j;\varphi'),\quad j=1,2, \tag{27}$$

where

$$w_j = C_{\mathrm{sca},j}/C_{\mathrm{ext},j},\quad j=1,2 \tag{28}$$

are the corresponding single scattering albedos, and

$$\tau_j = \int_0^x \mathrm{d}x'\,\rho(x')C_{\mathrm{ext},j}(x'),\quad j=1,2 \tag{29}$$

are the corresponding optical depths. Here, the extinction cross sections $C_{\mathrm{ext},1}$ and $C_{\mathrm{ext},2}$ are given by[14,15]

$$C_{\mathrm{ext},1} = \frac{4}{k}\,\mathrm{Re}\sum_{n=-\infty}^{\infty}\langle b_n\rangle, \tag{30}$$

$$C_{\mathrm{ext},2} = \frac{4}{k}\,\mathrm{Re}\sum_{n=-\infty}^{\infty}\langle a_n\rangle. \tag{31}$$

We now consider the light reflected by a homogeneous half-space of infinite cylinders. The reflection coefficients $\rho_j(\varphi,\varphi_0)$ and the backscattering coefficients $\sigma_j(\varphi_0)$ are defined by

$$\rho_j(\varphi,\varphi_0) = I_j(0;\pi-\varphi)/[I_j^0\cos\varphi_0], \tag{32}$$

$$\sigma_j(\varphi_0) = \rho_j(\varphi_0,\varphi_0)\cos\varphi_0,\quad j=1,2, \tag{33}$$

where $\varphi_0 \in [-\pi/2, \pi/2]$ is the angle of incidence, $\varphi \in [-\pi/2, \pi/2]$, and $I_j(0; \pi - \varphi)$ are the Stokes parameters of the reflected light. The reflection coefficients satisfy the nonlinear integral equation (see Chap. 30 of Ref. 10 and Refs. 18 and 4)

$$\begin{aligned}\rho_j(\varphi, \varphi_0)[\cos\varphi + \cos\varphi_0] = \frac{w_j}{2}\Bigg\{&\chi(\varphi + \varphi_0 + \pi) + \frac{\cos\varphi_0}{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi'\, \chi_j(\varphi - \varphi')\rho_j(\varphi', \varphi_0) \\ &+ \frac{\cos\varphi}{\pi}\int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi'\, \rho_j(\varphi, \varphi')\chi_j(\varphi' - \varphi_0) \\ &+ \frac{\cos\varphi\cos\varphi_0}{\pi^2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi' \int_{-\pi/2}^{\pi/2} \mathrm{d}\varphi''\, \rho_j(\varphi, \varphi') \\ &\times \chi_j(\varphi' + \varphi'' + \pi)\rho_j(\varphi'', \varphi_0)\Bigg\}, \quad j = 1, 2 \end{aligned} \tag{34}$$

and the symmetry relations[4]

$$\rho_j(\varphi, \varphi_0) = \rho_j(\varphi_0, \varphi) = \rho_j(-\varphi, -\varphi_0), \quad j = 1, 2. \tag{35}$$

To solve Eq. (34) numerically, we used the iterative scheme described in detail in Ref. 24 (see also Ref. 4).

Figure 2 shows illustrative computational data for the gamma distribution of cylinder radii with $r_{\mathrm{eff}} = 0.3\ \mu\mathrm{m}$ and $v_{\mathrm{eff}} = 0.05$. For comparison, analogous data are given in Fig. 3 for monodisperse Rayleigh cylinders with $r = 5 \times 10^{-4}\ \mu\mathrm{m}$. The corresponding single-scattering albedos are given in Table 3. We see that for the Rayleigh cylinders, $\sigma_2 > \sigma_1$ for all of the refractive indices and scattering angles. At the same time, for the size distribution of Mie cylinders, σ_1 is always greater than σ_2. The difference between the coefficients σ_1 and σ_2 for the Rayleigh cylinders decreases with an increase of absorption, while the difference for the Mie cylinders increases. It is also interesting to note that absorbing Rayleigh cylinders give greater backscattering coefficients than absorbing Mie cylinders with roughly the same single-scattering albedos. This fact can be easily understood as follows. The functions $\chi_1(\theta)$ and $\chi_2(\theta)$ for the Mie cylinders have large forward-scattering peaks and small backscattering peaks (see Table 2), while these functions for the Rayleigh cylinders have identical forward-scattering and backscattering peaks. Therefore, the photons diffusely reflected by the Mie cylinders have longer path lengths and undergo more scattering events than the photons diffusely reflected by the Rayleigh cylinders. As a result, the number of photons that are absorbed by the Mie cylinders is greater than the number of photons that are absorbed by Rayleigh cylinders with the same single-scattering albedos.

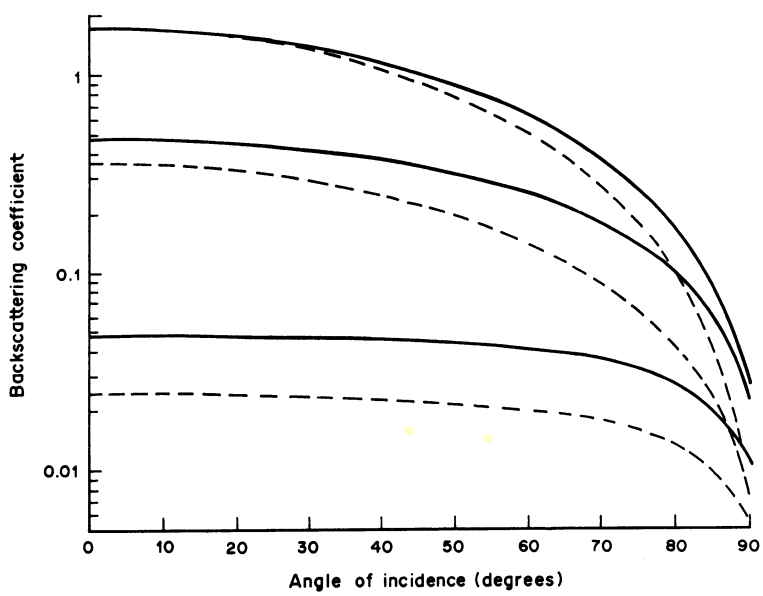

Fig. 2. Diffuse backscattering coefficients σ_1 (——) and σ_2 (– – –) vs the angle of incidence φ_0 for the gamma distribution of Mie cylinders with $r_{\mathrm{eff}} = 0.3\ \mu\mathrm{m}$ and $v_{\mathrm{eff}} = 0.05$. The real part of the refractive index is $\mathrm{Re}\, N = 1.32$ and the free-space wavelength is $\lambda = 0.5\ \mu\mathrm{m}$. The upper curves refer to $\mathrm{Im}\, N = 0$, the middle curves refer to $\mathrm{Im}\, N = 5 \times 10^{-3}$, and the lower curves refer to $\mathrm{Im}\, N = 5 \times 10^{-2}$.

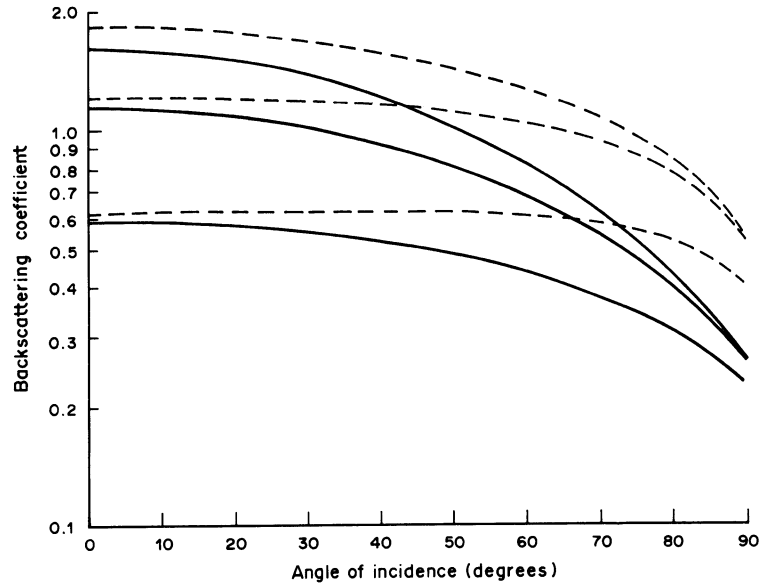

Fig. 3. The same as in Fig. 1, but for monodisperse Rayleigh cylinders with $r = 5 \times 10^{-4}\,\mu$m, Re $N = 1.32$, and $\lambda = 0.5\,\mu$m. The upper curves refer to Im $N = 0$, the middle curves refer to Im $N = 1 \times 10^{-7}$, and the lower curves refer to Im $N = 1 \times 10^{-6}$.

5. AN APPLICATION TO THE PROBLEM OF WEAK LOCALIZATION OF PHOTONS

Radiative transfer theory is widely used to calculate multiple scattering of light in discrete random media. Nevertheless, as was mentioned above, this theory is based on the ladder approximation of the Bethe–Salpeter equation and, therefore, cannot explain an interesting phenomenon, namely, enhanced backscattering of light from discrete disordered media (for recent reviews, see, e.g., Refs. 25 and 26). This phenomenon is associated with the so-called weak localization of photons and manifests itself as a well-defined narrow peak in the angular distribution of the intensity of the reflected light at scattering angles near 180°, which is superimposed on the diffusely reflected background intensity. This peak arises because a wave scattered through a certain multiple scattering path can interfere with the wave scattered through the time-reversed path, the interference being constructive in the backscattering direction.

The scalar theory of the enhanced backscattering from 3-D discrete random media is basically well understood (see, e.g., Refs. 27 and 28). While the diffusely-scattered background intensity comes from the sum of ladder diagrams, the coherent backscattering peak mainly comes from the sum of cyclical (or maximally crossed) diagrams (for terminology, see Refs. 29 and 30). A fundamental result of the scalar theory is that in exactly the backscattering direction, the contribution of all the cyclical diagrams to the backscattered intensity is identical to that of all the ladder diagrams of orders $n \geqslant 2$.[27] Therefore, the full backscattering coefficient γ is given by

$$\gamma = \gamma^{1} + \gamma^{L} + \gamma^{C} \equiv \gamma^{1} + 2\gamma^{L}, \tag{36}$$

Table 3. Single scattering albedos for the gamma distribution of Mie cylinders with $r_{\text{eff}} = 0.3\,\mu$m and $v_{\text{eff}} = 0.05$ and monodisperse Rayleigh cylinders with $r = 5 \times 10^{-4}\,\mu$m. The real part of the refractive index is Re $N = 1.32$ and the free-space wavelength is $\lambda = 0.5\,\mu$m.

Mie cylinders			Rayleigh cylinders		
Im N	w_1	w_2	Im N	w_1	w_2
$5 \cdot 10^{-3}$	0.973	0.973	$1 \cdot 10^{-7}$	0.985	0.970
$5 \cdot 10^{-2}$	0.789	0.783	$1 \cdot 10^{-6}$	0.866	0.764

where γ^1 is the contribution of the first-order scattering, γ^L the contribution of all the other ladder diagrams, and γ^C the contribution of all the cyclical diagrams. The sum $\gamma^1 + \gamma^L$ is identical to the common diffuse backscattering coefficient σ and can be found by solving the radiative transfer equation. An important characteristic of the coherent backscattering peak is the backscattering enhancement factor ζ, which is defined as the ratio of the total backscattered intensity to the incoherent background intensity in exactly the backscattering direction:

$$\zeta = [\gamma^1 + \gamma^L + \gamma^C]/[\gamma^1 + \gamma^L]. \tag{37}$$

From Eqs (36) and (37), we have

$$\zeta = \frac{\gamma^1 + 2\gamma^L}{\gamma^1 + \gamma^L} = \frac{2\sigma - \gamma^1}{\sigma} = 2 - \frac{\gamma^1}{\sigma}. \tag{38}$$

The general vector theory of weak localization of photons in 3-D discrete random media is very complicated and is still far from being completed. Nevertheless, as was pointed out here for the case of 2-D media, the vector problem is decomposed into two independent scalar subproblems, and Eqs. (36)–(38) hold for each of these subproblems. Thus, the corresponding backscattering enhancement factors ζ_1 and ζ_2 can easily be computed for particular scattering models. The results of such computations are shown in Table 4. The computations were performed for the gamma distribution of Mie cylinders with $N = 1.32$ and $N = 1.5$ and for conservative Rayleigh scattering with[3,5] $\chi_1(\theta) \equiv 1$, $\chi_2(\theta) = 1 + \cos 2\theta$, and $w_1 = w_2 = 1$. It follows from Eqs. (34) and (38) that the backscattering enhancement factors ζ_1 and ζ_2 obey the limits

$$\lim_{\varphi_0 \to \pi/2} \zeta(\varphi_0) = \lim_{w \to 0} \zeta(\varphi_0) = 1. \tag{39}$$

The first of these limits is demonstrated in Table 4. Also, we see that for the Mie cylinders, the backscattering enhancement factors are very close to 2 at angles of incidence near 0°. This well known "factor of two"[27] follows from the fact that for the (nearly) normal incidence and (nearly) conservative scattering, the contribution of the first-order scattering γ^1 is negligibly small compared to the total contribution of all orders of scattering σ [see Eq. (38)].

Table 4. Backscattering enhancement factors for the gamma distribution of Mie cylinders with $r_{eff} = 0.3\ \mu$m, $v_{eff} = 0.05$, and $\lambda = 0.5\ \mu$m and conservative Rayleigh scattering with $\chi_1(\theta) = 1$, $\chi_2(\theta) = 1 + \cos 2\theta$, and $w_1 = w_2 = 1$.

φ_0	Rayleigh scattering		Mie cylinders			
			$N = 1.32$		$N = 1.5$	
	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2
90°	1.0000	1.0000	1.000	1.000	1.000	1.000
89°	1.0780	1.0947	1.344	1.456	1.262	1.477
86°	1.2297	1.2430	1.672	1.792	1.538	1.806
83°	1.3365	1.3341	1.793	1.886	1.660	1.894
80°	1.4185	1.3993	1.855	1.928	1.732	1.932
70°	1.5949	1.5306	1.935	1.974	1.848	1.974
60°	1.6936	1.6023	1.961	1.986	1.897	1.985
50°	1.7540	1.6477	1.973	1.991	1.923	1.990
40°	1.7926	1.6786	1.979	1.993	1.938	1.993
30°	1.8174	1.7002	1.983	1.995	1.947	1.994
20°	1.8330	1.7146	1.985	1.995	1.953	1.995
10°	1.8415	1.7230	1.986	1.996	1.956	1.995
0°	1.8442	1.7257	1.986	1.996	1.957	1.995

Our value $\zeta_1(0) = 1.8442$ equals exactly that of Gorodnichev et al,[8,9] who considered enhanced backscattering of light from 2-D disordered media composed of isotropically scattering centers. In other words, these authors studied only the first of the two scalar Rayleigh subproblems with $\chi_1(\theta) \equiv 1$. It is interesting to note that from formulas of Ref. 5 and Eq. (38), one can easily derive an explicit analytical expression for the Rayleigh backscattering enhancement factor $\zeta_1(\varphi_0)$. This expression is

$$\zeta_1(\varphi_0) = 2 - \{(1 + \cos\varphi_0)\exp[2[f(\pi/2 + \varphi_0) + f(\pi/2 - \varphi_0)]/\pi]\}^{-1}, \tag{40}$$

where

$$f(x) = -\int_0^x \ln[2\sin(y/2)]\,\mathrm{d}y \tag{41}$$

is the Clausen integral.[31] For the particular case of $\varphi_0 = 0$,

$$\zeta_1(0) = 2 - \exp[-4\beta(2)/\pi]/2, \tag{42}$$

where $\beta(2) = f(\pi/2) = 0.91597\ldots$ is the Catalan constant.[31]

Note added in proof—Our discussion would be incomplete without citing Refs. 32–34, in which multiple scattering and localization of light in two-dimensional media was studied.

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